* PROBLEM 28.1

A) $\int \dfrac{1}{x^2 - 14x + 45 dx} \, dx$

$\int \dfrac{1}{(x-9)(x-5)} \, dx$

$\int \dfrac{A}{(x-9)} \, dx + \int \dfrac{B}{(x-5)} \, dx$

$\int \dfrac{\frac{1}{4}}{(x-9)} \, dx + \int \dfrac{-\frac{1}{4}}{(x-5)} \, dx$

$= \dfrac{1}{4} \log(x-9) - \dfrac{1}{4} \log(x-5) + C$

$= \boxed{\dfrac{\log(x-9) - \log(x-5)}{4} + C}$

$x^2 - 14x + 45 = (x-9)(x-5)$

$\dfrac{1}{(x-9)(x-5)} = \dfrac{A}{(x-9)} + \dfrac{B}{(x-5)}$

$\dfrac{1}{(x-9)(x-5)} = \dfrac{A(x-5) + B(x-9)}{(x-9)(x-5)}$

$1 = A(x-5) + B(x-9)$
$1 = Ax - 5A + Bx - 9B$
$1 = (A+B)x - 5A - 9B$

$\left| \begin{array}{l} 1 = -5A - 9B \\ 0 = A + B \end{array} \right|$ $\Rightarrow B = -A$

$1 = -5A + 9A$ $B = -A$
$1 = 4A$ $\boxed{B = -\dfrac{1}{4}}$
$\boxed{A = \dfrac{1}{4}}$

B) $\int \dfrac{2}{x^2 - 9} \, dx$

$\int \dfrac{2}{(x-3)(x+3)} \, dx$

$\int \dfrac{A}{(x-3)} \, dx + \int \dfrac{B}{(x+3)} \, dx$

$\int \dfrac{\frac{1}{3}}{(x-3)} \, dx + \int \dfrac{-\frac{1}{3}}{(x+3)} \, dx$

$= \dfrac{1}{3} \log(x-3) - \dfrac{1}{3} \log(x+3) + C$

$= \boxed{\dfrac{\log(x-3) - \log(x+3)}{3} + C}$

$x^2 - 9 = (x-3)(x+3)$

$\dfrac{2}{(x-3)(x+3)} = \dfrac{A}{x-3} + \dfrac{B}{x+3}$

$\dfrac{2}{(x-3)(x+3)} = \dfrac{A(x+3) + B(x-3)}{(x-3)(x+3)}$

$2 = A(x+3) + B(x-3)$
$2 = Ax + 3A + Bx - 3B$
$2 = (A+B)x + 3A - 3B$

$\left| \begin{array}{l} 2 = 3A - 3B \\ 0 = A + B \end{array} \right|$ $\Rightarrow A = -B$

$2 = 3A - 3B$ $B = -A$
$2 = 3A + 3A$ $\boxed{B = -\dfrac{1}{3}}$
$2 = 6A$
$\boxed{A = \dfrac{1}{3}}$

* PROBLEM 28.2

$$\int \frac{5}{4x^2 + 1} \, dx \qquad \begin{array}{l} u = 2x \\ du = 2dx \\ dx = \frac{du}{2} \end{array}$$

$$\frac{5}{2} \int \frac{1}{u^2 + 1} \, du$$

$$= \frac{5}{2} \arctan(u) + C$$

$$= \boxed{\frac{5 \arctan(2x)}{2} + C}$$

* PROBLEM 28.3

$$\int \frac{x^3 - x + 1}{x^2 - 1} \, dx \qquad\qquad \frac{1}{x^2 - 1} + x = \frac{1 + x^3 - x}{x^2 - 1}$$

$$\int \frac{1}{x^2 - 1} + x \, dx \qquad\qquad \frac{1}{(x+1)(x-1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)}$$

$$\int \frac{1}{(x-1)(x+1)} \, dx + \int x \, dx \qquad \text{Ⓐ} \lim_{x \to -1} \frac{1}{x-1} = A + \frac{B(x+1)}{(x-1)} \quad \Rightarrow A = -\frac{1}{2}$$

$$\int \frac{A}{(x+1)} \, dx + \int \frac{B}{(x-1)} \, dx + \int x \, dx \qquad \text{Ⓑ} \lim_{x \to 1} \frac{1}{x+1} = \frac{A(x-1)}{x+1} + B \quad \Rightarrow B = \frac{1}{2}$$

$$\int \frac{-\frac{1}{2}}{(x+1)} \, dx + \int \frac{\frac{1}{2}}{(x-1)} \, dx + \int x \, dx$$

$$= -\frac{\log(x+1)}{2} + \frac{\log(x-1)}{2} + \frac{x^2}{2} + C$$

$$= \boxed{\frac{\log(x-1) - \log(x+1) + x^2}{2} + C}$$

* PROBLEM 28.4

$$\int \frac{23}{(X+2)(X-3)(X-2)(X+3)} \, dx$$

$$23 \int \frac{1}{(X+2)(X-3)(X-2)(X+3)} \, dx$$

$$23 \left(\int \frac{A}{(X+2)} \, dx + \int \frac{B}{(X-3)} \, dx + \int \frac{C}{(X-2)} \, dx + \int \frac{D}{(X+3)} \, dx \right)$$

$$23 \left(\int \frac{\frac{1}{20}}{X+2} \, dx + \int \frac{\frac{1}{30}}{X-3} \, dx + \int \frac{-\frac{1}{20}}{X-2} \, dx + \int \frac{-\frac{1}{30}}{X+3} \, dx \right)$$

$$= 23 \left(\frac{1}{20} \log(X+2) + \frac{1}{30} \log(X-3) - \frac{1}{20} \log(X-2) - \frac{1}{30} \log(X+3) \right) + C$$

$$= 23 \left(\frac{\log(X+2) - \log(X-2)}{20} + \frac{\log(X-3) - \log(X+3)}{30} \right) + C$$

$$= \boxed{\frac{23 \log(X+2)}{20} + \frac{23 \log(X-3)}{30} - \frac{23 \log(X-2)}{20} - \frac{23 \log(X+3)}{30} + C}$$

$$\frac{1}{(X+2)(X-3)(X-2)(X+3)} = \frac{A}{(X+2)} + \frac{B}{(X-3)} + \frac{C}{(X-2)} + \frac{D}{(X+3)}$$

(A) $\lim\limits_{X \to -2} \dfrac{1}{(X-3)(X-2)(X+3)} = A + \dfrac{B(X+2)}{X-3} + \dfrac{C(X+2)}{X-2} + \dfrac{D(X+2)}{X+3} \Rightarrow \boxed{A = \frac{1}{20}}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \qquad\quad\; 0 \qquad\quad\; 0$

(B) $\lim\limits_{X \to 3} \dfrac{1}{(X+2)(X-2)(X+3)} = \dfrac{A(X-3)}{X+2} + B + \dfrac{C(X-3)}{(X-2)} + \dfrac{D(X-3)}{(X+3)} \Rightarrow \boxed{B = \frac{1}{30}}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0 \qquad\qquad\; 0 \qquad\qquad\; 0$

(C) $\lim\limits_{X \to 2} \dfrac{1}{(X+2)(X-3)(X+3)} = \dfrac{A(X-2)}{(X+2)} + \dfrac{B(X-2)}{(X-3)} + C + \dfrac{D(X-2)}{(X+3)} \Rightarrow \boxed{C = -\frac{1}{20}}$

$\qquad\qquad\quad 4 \quad -1 \quad\; 5 \qquad\qquad\quad 0 \qquad\qquad\; 0 \qquad\qquad\; 0$

(D) $\lim\limits_{X \to -3} \dfrac{1}{(X+2)(X-3)(X-2)} = \dfrac{A(X+3)}{(X+2)} + \dfrac{B(X+3)}{(X-3)} + \dfrac{C(X+3)}{(X-2)} + D \Rightarrow \boxed{D = -\frac{1}{30}}$

$\qquad\qquad\quad -1 \quad -6 \quad -5 \qquad\qquad 0 \qquad\qquad\; 0 \qquad\qquad\; 0$

✶ PROBLEM 28.5

$$\int \frac{1}{(x+1)(x-1)(x+7)(x-3)} \, dx$$

$$\int \frac{A}{(x+1)} \, dx + \int \frac{B}{(x-1)} \, dx + \int \frac{C}{(x+7)} \, dx + \int \frac{D}{(x-3)} \, dx$$

$$\int \frac{\frac{1}{48}}{x+1} \, dx + \int \frac{-\frac{1}{32}}{x-1} \, dx + \int \frac{-\frac{1}{480}}{x+7} \, dx + \int \frac{\frac{1}{80}}{x-3} \, dx$$

$$= \frac{\log(x+1)}{48} - \frac{\log(x-1)}{32} - \frac{\log(x+7)}{480} + \frac{\log(x-3)}{80} + C$$

$$= \boxed{\frac{10\log(x+1) - 15\log(x-1) - \log(x+7) + 6\log(x-3)}{480} + C}$$

$$\frac{1}{(x+1)(x-1)(x+7)(x-3)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x+7} + \frac{D}{x-3}$$

A.) $\displaystyle \lim_{x \to -1} \underset{-2 \quad 6 \quad -4}{\frac{1}{(x-1)(x+7)(x-3)}} = A + \underset{0}{\frac{B(x+1)}{x-1}} + \underset{0}{\frac{C(x+1)}{x+7}} + \underset{0}{\frac{D(x+1)}{x-3}}$ ⟹ $\boxed{A = \frac{1}{48}}$

B.) $\displaystyle \lim_{x \to 1} \underset{2 \quad 8 \quad -2}{\frac{1}{(x+1)(x+7)(x-3)}} = \underset{0}{\frac{A(x-1)}{x+1}} + B + \underset{0}{\frac{C(x-1)}{x+7}} + \underset{0}{\frac{D(x-1)}{x-3}}$ ⟹ $\boxed{B = -\frac{1}{32}}$

C.) $\displaystyle \lim_{x \to -7} \underset{-6 \quad -8 \quad -10}{\frac{1}{(x+1)(x-1)(x-3)}} = \underset{0}{\frac{A(x+7)}{x+1}} + \underset{0}{\frac{B(x+7)}{x-1}} + C + \underset{0}{\frac{D(x+7)}{x-3}}$ ⟹ $\boxed{C = -\frac{1}{480}}$

D.) $\displaystyle \lim_{x \to 3} \underset{4 \quad 2 \quad 10}{\frac{1}{(x+1)(x-1)(x+7)}} = \underset{0}{\frac{A(x-3)}{x+1}} + \underset{0}{\frac{B(x-3)}{x-1}} + \underset{0}{\frac{C(x-3)}{x+7}} + D$ ⟹ $\boxed{D = \frac{1}{80}}$