MATH 1A - PROBLEM SET 30 EVA GOLUZA

* PROBLEM 30.1

$f(x) = \frac{\cos(x)}{2}$ $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\Rightarrow$ PROBABILITY DENSITY FUNCTION

$\Rightarrow$ MEAN $= 0$

$$2M_2 = \int_{-\pi/2}^{\pi/2} x^2 \cos(x)\, dx$$

x^2	$\cos(x)$	
$2x$	$\sin(x)$	$+$
2	$-\cos(x)$	$-$
0	$-\sin(x)$	$+$

$$= x^2 \sin(x) + 2x\cos(x) - 2\sin(x) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= (x^2 - 2)\sin(x) + 2x\cos(x) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{\pi^2 - 8}{2}$$

$$= \boxed{\frac{\pi^2 - 8}{4} = 0.4674}$$

* PROBLEM 30.2

$f(x) = \frac{1}{b-a}$ $[a,b]$ $a \leq x \leq b$, else $= 0$ $a = 1$, $b = 5$ $f(x) = \frac{1}{5-1}$ $f(x) = \frac{1}{4}$

A)

$$M_n = \int_{-\infty}^{\infty} x^n f(x)\, dx$$

$$\int_{-\infty}^{\infty} x^n \left(\frac{1}{5-1}\right) dx$$

$$\int_{-\infty}^{\infty} \frac{x^n}{4}\, dx$$

$$\frac{1}{4}\int_{-\infty}^{\infty} x^n\, dx$$

$$= \frac{1}{4} \cdot \frac{x^{n+1}}{n+1}$$

$$= \boxed{\frac{x^{n+1}}{4(n+1)} \Big|_{-\infty}^{\infty}}$$

B)

VARIANCE

$$\mathrm{Var}(f) = M_2 - M_1^2$$

$$\int_1^5 \frac{x^2}{4}\, dx - \left(\int_1^5 \frac{x}{4}\, dx\right)^2$$

$$\int_1^5 \frac{x^2}{4}\, dx - 9$$

$$= \frac{31}{3} - 9$$

$$\boxed{= \frac{4}{3}}$$

STANDARD DIVIATION

$$\sigma = \sqrt{\mathrm{Var}(f)}$$

$$\sigma = \sqrt{\frac{4}{3}}$$

$$\sigma = \frac{2\sqrt{3}}{3}$$

$$\boxed{\sigma = 1.1547}$$

* PROBLEM 30.3

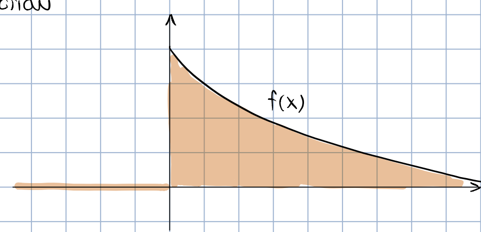
$f(x)$ for $x < 0$ $f(x) = 0$

for $x > 0$ $f(x) = \frac{1}{\log(2)} \frac{e^{-x}}{1 + e^{-x}}$

A) => VERIFYING THAT f(x) IS A PROBABILITY DENSITY FUNCTION

1. f(x) IS POSITIVE EVERYWHERE

2. f(x) IS A PIECE-WISE CONTINOUS FUNCTION



3. $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\int_0^{\infty} \frac{1}{\log(2)} \frac{e^{-x}}{1 + e^{-x}}\, dx$$

$$-\frac{1}{\log(2)} \int_0^{\infty} \frac{e^{-x}}{1 + e^{-x}}\, dx \qquad u = e^{-x} + 1, \quad du = -e^{-x}\, dx$$

$$-\frac{1}{\log(2)} \int_0^{\infty} \frac{1}{u}\, du$$

$$-\frac{1}{\log(2)} \cdot \log(u) + C$$

THIS IS ALSO A STEP TO FIND CDF (THE ANTIDERIVATIFE FOR PDF)

$$= -\frac{1}{\log(2)} \cdot \log(e^{-x} + 1) + C$$

$$= -\frac{\log(e^{-x} + 1)}{\log(2)} + C$$

=> ASSUMING THAT $F(x) = 0$ FOR $x \leq 0$, $F(0) = 0$

$$-\frac{\log(e^{(0)} + 1)}{\log(2)} + C = 0$$

$$-\frac{\log(2)}{\log(2)} + C = 0$$

$$\boxed{C = 1}$$

=> $\boxed{\text{CDF } F(x) = 1 - \frac{\log(e^{-x} + 1)}{\log(2)}}$

B) $m = M_1 \quad \int_0^{\infty} x\, f(x)\, dx = \boxed{1.18}$

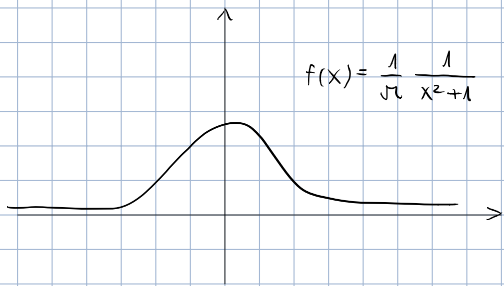
C) $M_2 \quad \int_0^{\infty} x^2 f(x)\, dx = \boxed{2.601}$

D) $\sigma = \sqrt{M_2 - M_1^2} = \boxed{1.19}$

* PROBLEM 30.4

A) PDF $f(x) = \frac{\frac{1}{\pi}}{x^2+1}$ VERIFY THAT CDF $f(y) = \frac{1}{2} + \frac{\arctan(y)}{\pi}$

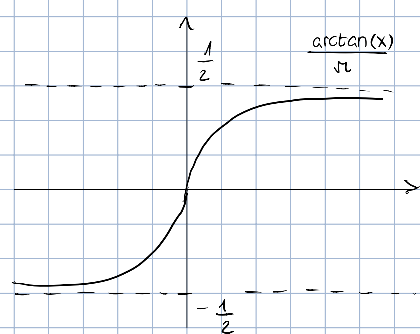
$$f(x) = \frac{1}{\pi}\,\frac{1}{x^2+1}$$



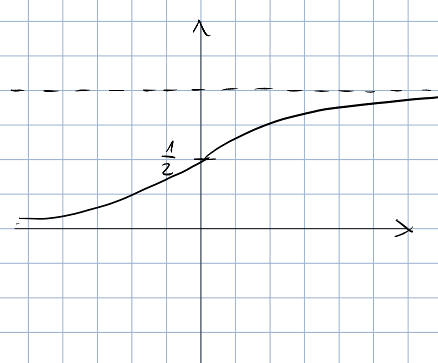
$$F(x) = \int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)}\,dx$$

$$= \frac{1}{\pi}\int \frac{1}{x^2+1}\,dx$$

$$= \frac{\arctan(x)}{\pi}$$



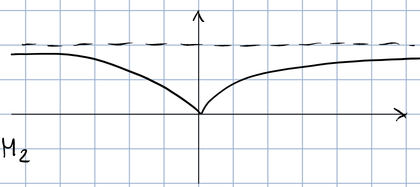
=> WE HAVE TO SHIFT f(x) UP
UNTIL IT BECOMES POSITIVE



$$\boxed{F(x) = \frac{1}{2} + \frac{\arctan(x)}{\pi}}$$

B) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{\pi(1+x^2)}\,dx$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x^2}{1+x^2}\,dx = M_2$$



$\boxed{\infty}$

* PROBLEM 30.5

$$F(x) = \frac{1}{2} + \frac{\arctan(x)}{\pi}$$

$y = \tan(\pi x)$ IS CAUCHY DISTRIBUTED
IF x IS UNIFORMLY
DISTRIBUTED ON $[0, 1]$

PROOF y IS IN $[a, b]$ MEANS

x IS IN $\left[\frac{\arctan(a)}{\pi}, \frac{\arctan(b)}{\pi}\right]$

WHAT IS THE PROBABILITY

$$\frac{\arctan b}{\pi} - \frac{\arctan a}{b} = F(b) - F(a)$$

$$\int_{-\infty}^{\infty} f(x)\,dx$$

=> I USED MATHEMATICA TO COMPUTE 10 RANDOM CAUCHY DISTRIBUTED NUMBERS