

4/8/2020: First hourly Practice B

Your Name:

- Solutions are submitted as PDF handwritten in a file called after your name. Capitalize the first letters like OliverKnill.pdf. Hand-write your paper. It can be handwritten also electronically using an ipad but it needs to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False \cdots 20: False. Also, please sign your solutions.
- No books, calculators, computers, or other electronic aids are allowed. You can use one page of your own handwritten notes when writing the paper.
- The exam will be released at 9 AM on April 8. Try to do it during regular class time. You have to submit the paper within 24 hours by emailing it to knill@math.harvard.edu as an attachment. Submit it soon so that technical problems can be ironed out. It is your responsibility to submit the paper on time and get within that time also a confirmation.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

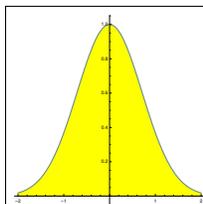
Problem 1) TF questions (20 points) No justifications are needed.

- 1) T F If f is a continuous function then $\int_0^x f(t) dt$ is an area and therefore positive.
- 2) T F The anti-derivative of $\operatorname{arccot}(x)$ is $-\log(\sin(x)) + C$.
- 3) T F The fundamental theorem of calculus implies that $\int_0^3 f''(x) dx = f'(3) - f'(0)$.
- 4) T F The volume of a cylinder of height 3 and radius 5 is given by the integral $\int_0^3 \pi 5^2 dx$.
- 5) T F The anti-derivative of $\tan(x)$ is $1/\cos^2(x)$.
- 6) T F The mean value theorem implies that the derivative of $\sin(x)$ in the interval $[0, \pi/2]$ is $2/\pi$ somewhere.
- 7) T F The function $F(x) = \int_0^x \sin(t^2) dt$ has the derivative $\sin(x^2)$.
- 8) T F If f is a PDF, then $\int_{-\infty}^{\infty} xf(x) dx$ is called the mean.
- 9) T F The identity $\frac{d}{dx} \int_0^1 \sin(x) dx = \sin(1)$ holds.
- 10) T F If a solid is scaled by a factor 2 in all directions then its volume increases by a factor 8.
- 11) T F If f is a PDF, then $\int_0^2 f(x) dx$ is the probability that the data are in $[0, 2]$.
- 12) T F If $f(x)$ is smaller than $g(x)$ for all x , then $\int_0^1 f(x) - g(x) dx$ is negative.
- 13) T F Every improper integral defines an infinite area.
- 14) T F The anti derivative of $f'(x)$ is equal to $f(x) + c$.
- 15) T F Catastrophes can explain why minima can change discontinuously.
- 16) T F If f is discontinuous at 0, then $\int_{-1}^1 f(x) dx$ is infinite.
- 17) T F If $f(-\infty) = 0$ and $f(\infty) = 1$ then $f' = 1$ somewhere on $(-\infty, \infty)$.
- 18) T F The anti-derivative of $1/x$ is $\log(x) + C$, where \log is the natural log.
- 19) T F A catastrophe is defined as a critical point of f which is a minimum.
- 20) T F The integral $\int_0^{\infty} 1/x^2 dx$ represents a finite area.

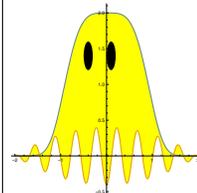
Problem 2) Matching problem (10 points) No justifications are needed.

a) (6 points) Match the following integrals with the regions. Graphs 1) and 2) are inspired by a cartoon by Matthew Freeman (J Epidemiol. Community Health. 2006 January; 60(1): 6)

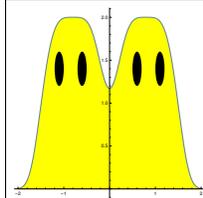
Integral	Fill in 1-4
$\int_{-2}^2 (4 - x^2) \cos^2(14x)/10 - (4 - x^2) \cos(14x)/15 dx$	
$\int_{-2}^2 2 \exp(-3(x + 0.8)^4) + 2 \exp(-3(x - 0.8)^4) dx$	
$\int_{-2}^2 \exp(-x^2) dx$	
$\int_{-2}^2 2 \exp(-x^4) - (x^2 - 4) \cos(14x)/10 dx$	



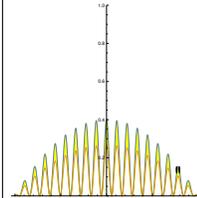
1) Normal distribution



2) Paranormal distribution



3) Abnormal distribution



4) Wormal distribution

b) (4 points) Which of the following statements follows from Rolle's theorem? Check only one.

Result	Check
If $f(0) = -1$ and $f(1) = 1$ then there is x with $0 \leq x \leq 1$ with $f'(x) = 2$	
If $f(0) = 1$ and $f(1) = 1$ then there is a critical point x of f in $(0, 1)$	
If $f(0) = 1$ and $f(1) = 1$ then there is point where $f(x) = 2$ in $(0, 1)$	
If $f(0) = 1$ and $f(1) = 1$ then there is point where $f''(p) = 0$ in $(0, 1)$	

Problem 3) (10 points)

a) (4 points) Having seen some applications of integration and differentiation, complete the table:

Function f	Antiderivative F
Probability density function	
	Total cost
	Mass
Area	
	Velocity
Power	
Velocity	

b) (2 points) We have seen two methods to find roots $f(x) = 0$ of equations. Both methods need some assumptions on the functions: Choose from the following: "differentiability", "continuity", "positivity".

Method	Assumption which f has to satisfy
Dissection method	
Newton method	

c) (2 points) Which is more general? In each row, check one box.

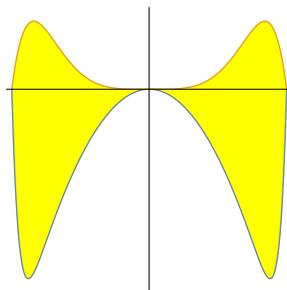
	Related rates	Implicit differentiation	
	Rolles theorem	Intermediate value theorem	

d) (2 points) Which integral is finite? Chose one!

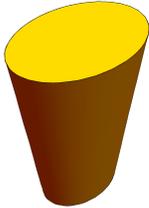
Integral	finite	infinite	
$\int_1^\infty 1/\sqrt{x} dx$			
$\int_1^\infty 1/x^2 dx$			

Problem 4) Area computation (10 points)

The region enclosed by the graphs of $f(x) = x^{20} - x^2$ and $g(x) = x^4 - x^8$ is a cross section for a catamaran sailing boat. Find the area.

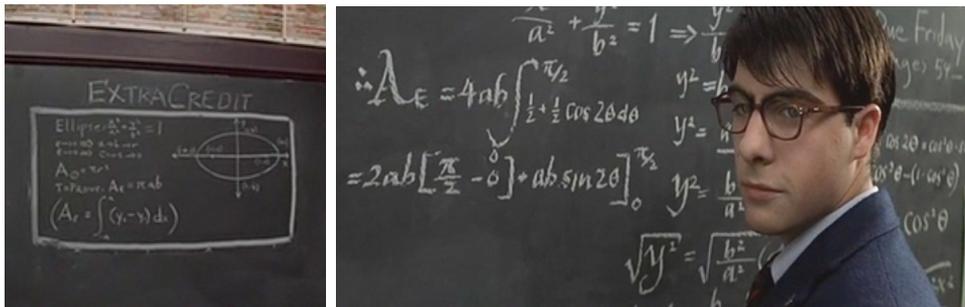


Problem 5) Volume computation (10 points)



We are given that an ellipse with diameters $2b$ and $2a$ has area πab . Find the volume of part of a cone whose height is between $z = 3$ and $z = 5$ for which the cross section at height z is an ellipse with parameters $a = 2z$ and $b = 3z$.

Remark. We will see later the area formula. In the movie “**Rushmore**”, the teacher tells about the problem: “I put that up as a joke. It’s probably the hardest geometry equation in the world”.



Screen shots from the movie Rushmore shows a blackboard where the formula for the ell.pdf is computed using trig substitution. You might spot a double angle formula.

We will come to that.

Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals. Each of the problems produces a numerical answer.

a) (2 points) $\int_0^1 (x - 1)^4 dx$

b) (2 points) $\int_0^1 x^{1/3} dx$.

c) (2 points) $\int_0^{\sqrt{3}} \frac{6}{1+x^2} dx$

d) (2 points) $\int_{-2}^{e-3} \frac{5}{3+x} dx$

e) (2 points) $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$.

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

a) (2 points) $\int e^{7x} - \sqrt{x} dx$

b) (2 points) $\int \frac{5}{x+1} + 7 \cos^2(x) dx$

c) (2 points) $\int \frac{11}{1+x^2} + 9 \tan(x) dx$

d) (2 points) $\int \frac{4}{\cos^2(x)} + \frac{2}{\sin^2(x)} dx$

e) (2 points) $\int 2x \cos(x^2) dx$

Problem 8) PDF's and CDF's (10 points)

Verify that the function f which is defined to be $f(x) = 0$ for $x < 1$ and $f(x) = 3/x^4$ on $[1, \infty)$ is a PDF.

Problem 9) Catastrophes (10 points)

We look at the one-parameter family of functions $f_c(x) = x^6 - cx^4 - cx^2$, where c is a parameter.

a) (4 points) Verify that f has a critical point 0 for all c .

b) (3 points) Determine whether 0 is a minimum or maximum depending on c .

c) (3 points) For which c does a catastrophe occur?

Problem 10) Basic integrals (10 points)

Find the anti derivatives. You have to solve in 10 seconds each. For every second over that limit, one point of the entire exam will be taken off. So, for example: if you use 62 seconds for the following 5 problems, you have used 12 seconds too much and 12 points are taken off from your exam. Don't worry, we do not assign negative points; your final score will always remain a number between 0 and 110 points. To fill a loophole in that setup: if you choose not do the problems, 50 points are taken off.

a) (2 points) e^{-2x} .

b) (2 points) $\cos(15x)$.

c) (2 points) 2^x .

d) (2 points) $1/(1 - x)$

e) (2 points) $1/(1 + x^2)$