Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2020

4/8/2020: First hourly Practice C

Your Name:

- Solutions are submitted as PDF handwritten in a file called after your name. Capitalize the first letters like OliverKnill.pdf. Hand-write your paper. It can be handwritten also electronically using an ipad but it needs to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ 20: False. Also, please sign your solutions.
- No books, calculators, computers, or other electronic aids are allowed. You can use one page of your own handwritten notes when writing the paper.
- The exam will be released at 9 AM on April 8. Try to do it during regular class time. You have to submit the paper within 24 hours by emailing it to knill@math.harvard.edu as an attachment. Submit it soon so that technical problems can be ironed out. It is your responsibility to submit the paper on time and get within that time also a confirmation.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F The anti-derivative of $\tan(x)$ is $-\log(\cos(x)) + C$.

2) T F The fundamental theorem of calculus implies that $\int_0^1 f'(x)\,dx = f(1) - f(0)$.

3) T F The volume of truncated pyramid with a base square length 2 and top square length 3 is given by the integral $\int_2^3 x^2\,dx$.

4) T F The derivative of $\arctan(x)$ is $1/\cos^2(x)$.

5) T F The mean value theorem implies $\int_a^b f'(x)\,dx = f'(c)(b-a)$ for some c in the interval (a,b).

6) T F If $F(x) = \int_0^x f(t)\,dt$ has an critical point at $x = 1$ then f has a root at $x = 1$.

7) T F The anti-derivative of the derivative of f is equal to $f + C$ where C is a constant.

8) T F The CDF is an anti-derivative of a PDF.

9) T F The identity $\frac{d}{dx}\int_5^9 f(x)\,dx = f(9) - f(5)$ holds for all continuous functions f.

10) T F Two surfaces of revolution which have the same cross section area $A(x)$ also have the same volume.

11) T F The integral $\int_{-\infty}^0 e^x\,dx$ is an improper integral which converges.

12) T F The identity $\int_2^9 7f(x)\,dx = 7\int_2^9 f(x)\,dx$ is true for all continuous functions f.

13) T F The improper integral $\int_1^\infty 1/x\,dx$ in the sense that $\int_1^R 1/x\,dx$ converges for $R \to \infty$ to a finite value.

14) T F If $f_c(x)$ has a local minimum at $x = 2$ for $c < 1$ and no local minimum anywhere for $c > 1$, then $c = 1$ is a catastrophe.

15) T F An improper integral is an indefinite integral which does not converge.

16) T F If $f(-5) = 0$ and $f(5) = 10$ then $f' = 1$ somewhere on the interval $[-5, 5]$.

17) T F The sum $\frac{1}{n}\sum_{k=0}^{n-1}\frac{k}{n} = \frac{1}{n}[\frac{0}{n} + \frac{1}{n} + \cdots + \frac{n-1}{n}]$ is a Riemann sum to the integral $\int_0^1 x\,dx$.

18) T F The anti-derivative of $\text{sinc}(x) = \sin(x)/x$ is equal to $\sin(\log(x)) + C$.

19) T F The anti-derivative of $\log(x)$ is $1/x + C$.

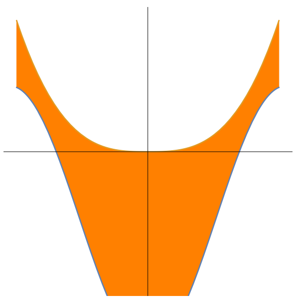
20) T F We have $\int_0^x t f(t)\,dt = x\int_0^x f(t)\,dt$ for all functions f.

Problem 2) Matching problem (10 points) No justifications are needed.

a) (6 points) Match the integrals with the pictures.

Integral	Enter 1-6
$\int_{-1}^{1}(1-x)^2\ dx$	
$\int_{-1}^{1}\lvert x\rvert\ dx$	
$\int_{-1}^{1} x^4\ dx$	

Integral	Enter 1-6
$\int_{-1}^{1}\lvert x\rvert^3-\cos(3x)\ dx$	
$\int_{-1}^{1}[\sin^2(\pi x)-\cos^2(\pi x)]\ dx$	
$\int_{-1}^{1} 1-\lvert x\rvert\ dx$	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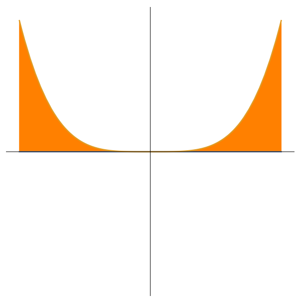
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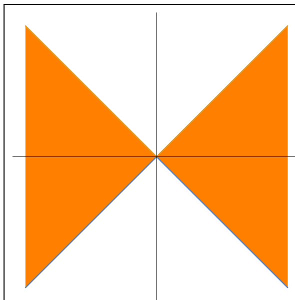
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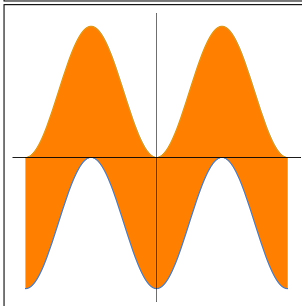
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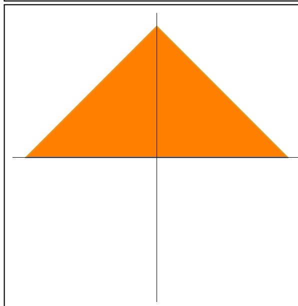
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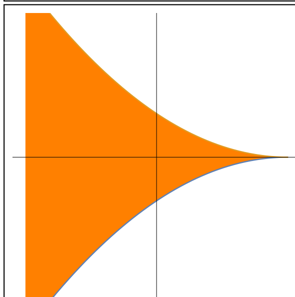
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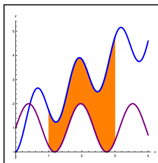
b) (4 points) Match the concepts: each of the 4 figures illustrates one of the formulas which are the centers of the **mind map** we have drawn for this exam: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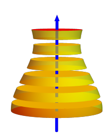
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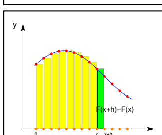
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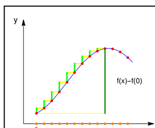

4)

Formula	Enter 1-4
$\int_a^b A(z)\ dz$	
$\int_a^b g(x)-f(x)\ dx$	

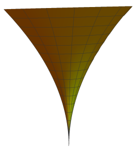
Formula	Enter 1-4
$\frac{d}{dx}\int_0^x f(t)\ dt = f(x)$	
$\int_0^x f'(t)\ dt = f(x)-f(0)$	

Problem 3) Matching problem (10 points) No justifications are needed.

a) (6 points) Match the volumes of solids.

Integral	Enter 1-6
$\int_0^1 \pi z^4 \, dz$	
$\int_0^1 \pi z \, dz$	
$\int_0^1 \pi(4+\sin(4z)) \, dz$	

Integral	Enter 1-6
$\int_{-1}^1 \pi e^{-4z^2} \, dz$	
$\int_0^1 \pi z^2 \, dz$	
$\int_0^1 (1-z)^2 \, dz$	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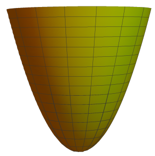
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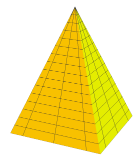
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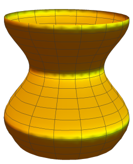
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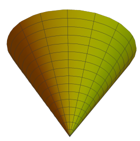
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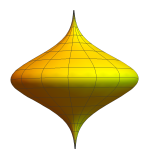

6)

b) (4 points) Fill in the missing word which links **applications** of integration.

The probability density function is the		of the cumulative distribution function.
The total cost is the		of the marginal cost.
The volume of a solid is the		of the cross section area function.
The velocity of a ball is the		of the acceleration of the ball.

Problem 4) Area computation (10 points)

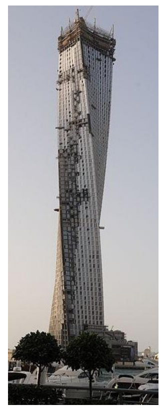
Find the area of the region enclosed the graphs of $y = x^4 - 12$ and $y = 8 - x^2$.

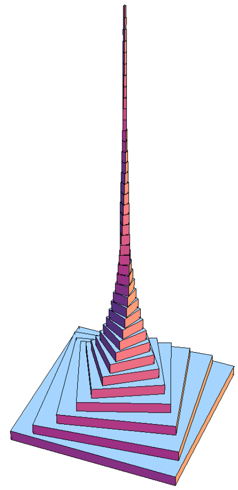
Problem 5) Volume computation (10 points)

The **infinity tower** in Dubai of height 330 meters has floors which can rotate. After much delay, it is expected to be completed this year. Inspired by the name "infinity", we build a new but twisted science center for which the side length of the square floor is

$$l(z) = \frac{1}{1+z} \; .$$

Find the volume of this new **Harvard needle building** which extends from 0 to ∞. We are the best!





Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals. You should get a definite real number in each case.

a) (2 points) $\int_0^\infty e^{-x} \, dx$

b) (3 points) $\int_0^1 x^{1/5} + x^3 \, dx$.

c) (3 points) $\int_{-1}^1 \frac{1}{1+x^2} \, dx$

d) (2 points) $\int_0^{e-1} \frac{2}{1+x} \, dx$

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

a) (2 points) $\int \frac{3}{\sqrt{1+3x}} + \cos(x) \, dx$

b) (3 points) $\int e^{x/5} - 7x^6 + \frac{4}{x^2+1} \, dx$

c) (2 points) $\int \frac{4}{e^{4x+5}} + 3\sin(x) \, dx$

d) (3 points) $\int \frac{1}{\sin^2(x)} + \frac{4}{x} \, dx$

Problem 8) PDF's and CDF's (10 points)

In order to verify that if f is a PDF and g is a PDF, then $(f+g)/2$ is a PDF, which three conditions do we have to check?

a) The function is piecewise [ ] b) The function is [ ]

c) The integral [ ] is equal to [ ].

Problem 9) Catastrophes (10 points)

Verify first for each of the following functions that $x = 0$ is a critical point. Then give a criterium for stability of $x = 0$. The answer will depend on c.

a) (3 points) $f(x) = x^5 + 2x^2 - cx^2$.

b) (3 points) $f(x) = x^4 + cx^2 - x^2$.

Determine now in both examples for which parameter c the catastrophe occurs

c) (2 points) in the case $f(x) = x^5 + 2x^2 - cx^2$.

d) (2 points) in the case $f(x) = x^4 + cx^2 - x^2$.