Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2020

4/8/2020: First hourly Practice B

Your Name:

- Solutions are submitted as PDF handwritten in a file called after your name. Capitalize the first letters like OliverKnill.pdf. Hand-write your paper. It can be handwritten also electronically using an ipad but it needs to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False ··· 20: False. Also, please sign your solutions.
- No books, calculators, computers, or other electronic aids are allowed. You can use one page of your own handwritten notes when writing the paper.
- The exam will be released at 9 AM on April 8. Try to do it during regular class time. You have to submit the paper within 24 hours by emailing it to knill@math.harvard.edu as an attachment. Submit it soon so that technical problems can be ironed out. It is your responsibility to submit the paper on time and get within that time also a confirmation.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T **F** If f is a continuous function then $\int_0^x f(t)\, dt$ is an area and therefore positive.

Solution:
Take $f(x) = -x$ then $\int_0^x -t\ dt = -x^2/2$ is negative. Parts under the x axes contribute negatively to the integral.

2) T **F** The anti-derivative of $\mathrm{arccot}(x)$ is $-\log(\sin(x)) + C$.

Solution:
Differentiate the right hand side to check.

3) T **F** The fundamental theorem of calculus implies that $\int_0^3 f''(x)\ dx = f'(3) - f(0)$.

Solution:
Yes this is a special case of the fundamental theorem, if there were not the missing prime at the end. It had been noted on the blackboard that there is no typo in this problem.

4) **T** F The volume of a cylinder of height 3 and radius 5 is given by the integral $\int_0^3 \pi 5^2\ dx$.

Solution:
Yes the area of a slice is $r^2\pi$.

5) T **F** The anti-derivative of $\tan(x)$ is $1/\cos^2(x)$.

Solution:
The derivative of $\tan(x)$ is $1/\cos(x)^2$.

6) **T** F The mean value theorem implies that the derivative of $\sin(x)$ in the interval $[0, \pi/2]$ is $2/\pi$ somewhere.

Solution:
This is a typical application of the mean value theorem.

7) T F The function $F(x) = \int_0^x \sin(t^2)\, dt$ has the derivative $\sin(x^2)$.

Solution:
The first derivative of F is f.

8) T F If f is a PDF, then $\int_{-\infty}^{\infty} x f(x)\, dx$ is called the mean.

Solution:
Yes, this is the definition.

9) T F The identity $\frac{d}{dx} \int_0^1 \sin(x)\, dx = \sin(1)$ holds.

Solution:
We differentiate a constant. This was the most commonly wrongly checked problem.

10) T F If a solid is scaled by a factor 2 in all directions then its volume increases by a factor 8.

Solution:
Yes, the volume goes cubic.

11) T F If f is a PDF, then $\int_0^2 f(x)\, dx$ is the probability that the data are in $[0, 2]$.

Solution:
Yes, we have defined it like that.

12) T F If $f(x)$ is smaller than $g(x)$ for all x, then $\int_0^1 f(x) - g(x)\, dx$ is negative.

Solution:
Yes, we can take the 7 constant outside the integral.

13) T **F** Every improper integral defines an infinite area.

Solution:
No, it can be finite.

14) **T** F The anti derivative of $f'(x)$ is equal to $f(x) + c$.

Solution:
Yes, taking anti derivatives cancels taking derivatives, up to a constant.

15) **T** F Catastrophes can explain why minima can change discontinuously.

Solution:
This is a definition.

16) T **F** If f is discontinuous at 0, then $\int_{-1}^{1} f(x)\, dx$ is infinite.

Solution:
Think about the sign function. The integral is finite.

17) T **F** If $f(-\infty) = 0$ and $f(\infty) = 1$ then $f' = 1$ somewhere on $(-\infty, \infty)$.

Solution:
No, the slope can be arbitrarily small. We have only to increase the value by 1 and all the space of the world to do that. This was probably the second most commonly wrongly checked TF problem in this exam.

18) T F The anti-derivative of $1/x$ is $\log(x) + C$, where log is the natural log.

Solution:
Yes, we know that, and should never,never,never,never forget!

19) T F A catastrophe is defined as a critical point of f which is a minimum.

Solution:
No, it deals with changes of critical points.

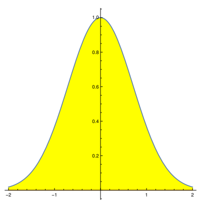
20) T F The integral $\int_0^\infty 1/x^2 \, dx$ represents a finite area.

Solution:
We have no problem at infinity but a problem at $x = 0$.

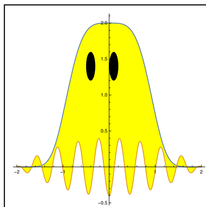
Problem 2) Matching problem (10 points) No justifications are needed.

a) (6 points) Match the following integrals with the regions. Graphs 1) and 2) are inspired by a cartoon by Matthew Freeman (J Epidemiol. Community Health. 2006 January; 60(1): 6)

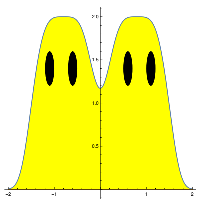
Integral	Fill in 1-4
$\int_{-2}^{2}(4-x^2)\cos^2(14x)/10-(4-x^2)\cos(14x)/15dx$	
$\int_{-2}^{2} 2\exp(-3(x+0.8)^4)+2\exp(-3(x-0.8)^4\ dx$	
$\int_{-2}^{2} \exp(-x^2)\ dx$	
$\int_{-2}^{2} 2\exp(-x^4)-(x^2-4)\cos(14x)/10\ dx$	



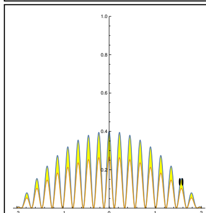
1) Normal distribution



2) Paranormal distribution



3) Abnormal distribution



4) Wormal distribution

b) (4 points) Which of the following statements follows from Rolle's theorem? Check only one.

Result	Check
If $f(0)=-1$ and $f(1)=1$ then there is x with $0\leq x\leq 1$ with $f'(x)=2$	
If $f(0)=1$ and $f(1)=1$ then there is a critical point x of f in $(0,1)$	
If $f(0)=1$ and $f(1)=1$ then there is point where $f(x)=2$ in $(0,1)$	
If $f(0)=1$ and $f(1)=1$ then there is point where $f''(p)=0$ in $(0,1)$	

Solution:
a) 4,3,1,2.
b) second choice.

Problem 3) (10 points)

a) (4 points) Having seen some applications of integration and differentiation, complete the table:

Function f	Antiderivative F
Probability density function	
	Total cost
	Mass
Area	
	Velocity
Power	
Velocity	

b) (2 points) We have seen two methods to find roots $f(x) = 0$ of equations. Both methods need some assumptions on the functions: Choose from the following: "differentiability", "continuity", "positivity".

Method	Assumption which f has to satisfy
Disection method	
Newton method	

c) (2 points) Which is more general? In each row, check one box.

	Related rates	Implicit differentiation	
	Rolles theorem	Intermediate value theorem	

d) (2 points) Which integral is finite? Chose one!

Integral	finite	infinite	
$\int_1^\infty 1/\sqrt{x}\ dx$			
$\int_1^\infty 1/x^2\ dx$			

Solution:

Function f	Antiderivative F
Probability density function	CDF
Marginal cost	Total cost
Density	Mass
Area	Volume
Acceleration	Velocity
Power	Work or Energy
Velocity	Position

b) (2 points) We have seen two methods to find roots $f(x) = 0$ of equations. Both methods need some assumptions on the functions: Choose from the following: "differentiability", "continuity", "positivity".

Method	Assumption which f has to satisfy
Disection method	Continuity
Newton method	Differnentiability

c) (2 points) Which is more general? In each row, check one box.

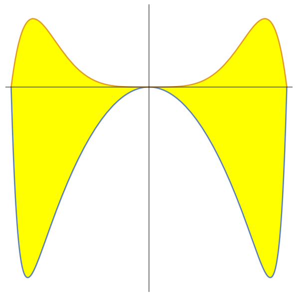
X	Related rates	Implicit differentiation	
	Rolles theorem	Intermediate value theorem	X

d) (2 points) Which integral is finite? Chose one!

Integral	finite	infinite
$\int_1^\infty 1/\sqrt{x}\, dx$		X
$\int_1^\infty 1/x^2\, dx$	X	

Problem 4) Area computation (10 points)

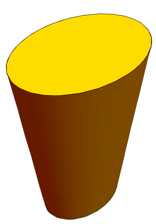
The region enclosed by the graphs of $f(x) = x^{20} - x^2$ and $g(x) = x^4 - x^8$ is a cross section for a catamaran sailing boat. Find the area.



Solution:
Note that $x^4 - x^8$ is positive on $[-1,1]$ an d $x^{20} - x^2$ is negative:

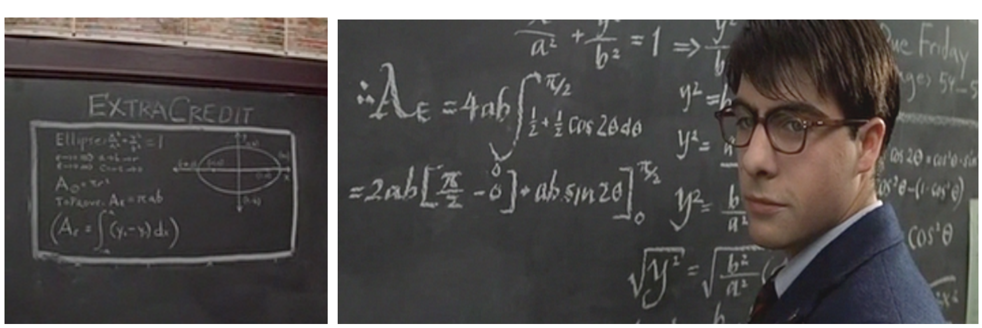
$$\int_{-1}^{1} x^4 - x^8 - (x^{20} - x^2)\, dx = 8/45 + 4/7 = 236/315 \, .$$

Problem 5) Volume computation (10 points)



We are given that an ellipse with diameters $2b$ and $2a$ has area πab. Find the volume of part of a cone whose height is between $z = 3$ and $z = 5$ for which the cross section at height z is an ellipse with parameters $a = 2z$ and $b = 3z$.

Remark. We will see later the area formula. In the movie **"Rushmore"**, the teacher tells about the problem: "I put that up as a joke. It's probably the hardest geometry equation in the world".



Screen shots from the movie Rushmore shows a blackboard where the formula for the ell.pdfe is computed using trig substitution. You might spot a double angle formula. We will come to that.

Solution:
The area at height z is $\pi 6z^2$. The answer is $\int_3^5 \pi 6z^2 \, dx = 2\pi z^3|_3^5 = 196\pi$.

Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals. Each of the problems produces a numerical answer.

a) (2 points) $\int_0^1 (x-1)^4 \, dx$

b) (2 points) $\int_0^1 x^{1/3} \, dx$.

c) (2 points) $\int_0^{\sqrt{3}} \frac{6}{1+x^2} \, dx$

d) (2 points) $\int_{-2}^{e-3} \frac{5}{3+x} \, dx$

e) (2 points) $\int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx$.

Solution:
a) 5^{-1}
b) $3/4$
c) 2π
d) 5
e) $\pi/2$

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

a) (2 points) $\int e^{7x} - \sqrt{x} \, dx$

b) (2 points) $\int \frac{5}{x+1} + 7\cos^2(x) \, dx$

c) (2 points) $\int \frac{11}{1+x^2} + 9\tan(x) \, dx$

d) (2 points) $\int \frac{4}{\cos^2(x)} + \frac{2}{\sin^2(x)} \, dx$

e) (2 points) $\int 2x\cos(x^2) \, dx$

Solution:
a) $\exp(7x)/7 - 2x^{3/2}/3 + C$.
b) $7x/2 + 5\log(1+x) + 7\sin(2x)/4 + C$.
c) $11\arctan(x) - 9\log(\cos(x)) + C$.
d) $4\tan(x) - 2\cot(x) + C$.
e) $\sin(x^2) + C$.

Problem 8) PDF's and CDF's (10 points)

Verify that the function f which is defined to be $f(x) = 0$ for $x < 1$ and $f(x) = 3/x^4$ on $[1, \infty)$ is a PDF.

Solution:
It is piecewise continuous, is non-negative and satisfies $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

Problem 9) Catastrophes (10 points)

We look at the one-parameter family of functions $f_c(x) = x^6 - cx^4 - cx^2$, where c is a parameter.

a) (4 points) Verify that f has a critical point 0 for all c.

b) (3 points) Determine whether 0 is a minimum or maximum depending on c.

c) (3 points) For which c does a catastrophe occur?

Solution:
a) Differentiate to see that $f' = 6x^5 - 4x^3c - 2xc$ has critical points at $x = 0$.
b) The second derivative is $f''(x) = 30x^4 - 12x^2c - 2c$ which is $f''(0) = -2c$. We see that for $c < 0$ we have a minimum and for $c > 0$ we have a maximum.
c) $c = 0$ is the catastrophe, because for this parameter the number a minimum becomes a maximum.

Problem 10) Basic integrals (10 points)

Find the anti derivatives. You have to solve in 10 seconds each. For every second over that limit, one point of the entire exam will be taken off. So, for example: if you use 62 seconds for the following 5 problems, you have used 12 seconds too much and 12 points are taken off from your exam. Don't worry, we do not assign negative points; your final score will always remain a number between 0 and 110 points. To fill a loophole in that setup: if you choose not do the problems, 50 points are taken off.

a) (2 points) e^{-2x}.
b) (2 points) $\cos(15x)$.
c) (2 points) 2^x.
d) (2 points) $1/(1-x)$
e) (2 points) $1/(1+x^2)$

Solution:
The optimal strategy is of course to write down something within 10 seconds, even if you are not sure. But you should not have to think: these are all integrals you have to be able to do in your sleep:
a) $e^{-2x}/(-2) + c$,
b) $\sin(15x)/15 + c$,
c) Since this is $e^{x \log(2)}$, the answer is $2^x/\log(2) + c$
d) $-\log(1-x) + c$.
e) $\arctan(x) + c$.