Data project 2: Polyhedra

2.1. In the context of **data science**, **graphs** are very important. **Graph databases** started to appear already with the **Incas** in the form of **Khipu** documents. Nature has used DNA as a database since millions of years. In modern information technology, graph databases replace more and more traditional relational databases. A **graph** is a collection of nodes called **vertices** joined by connections called **edges**. Street maps, electric grids, directories, family or collaboration trees, brains, social networks or polyhedra are examples of graphs.

2.2. A graph G comes with a **graph polynomial**. This brings us to calculus. If the graph has a_0 vertices and a_1 edges and a_2 triangles the polynomial is defined as $f_G(x) = 1 + a_0 x + a_1 x^2 + a_2 x^3$. More generally, if a_k is the number of k-**cliques**, groups of k vertices all connected to each other, then $f_G(x) = 1 + \sum_{k=0} a_k x^{k+1}$, where Σ is the summation sign.
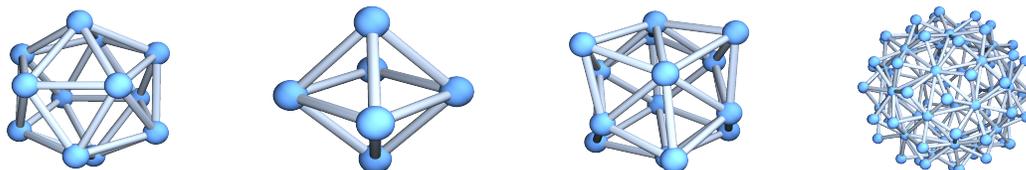

FIGURE 1. The Icosahedron, Octahedron, Tetraxishexahedron and Echidnahedron.

2.3. The **unit sphere** $S(v)$ of a vertex v consists of the subgraph build by all vertices connected to the node. If G is a friendship graph, and v is you, then $S(v)$ is the friend network of your friends without you. Let $f_{S(v)}(x)$ denote the polynomial function to $S(v)$.

Theorem: $f'_G(x) = \sum_{v \in V} f_{S(v)}(x)$

It is a **Gauss-Bonnet theorem** [1] because it allows to compute the Euler characteristic $\chi(G) = a_0 - a_1 + a_2 - \cdots$ of a graph by integrating and evaluating at -1: Indeed, we have from the **fundamental theorem of calculus** $\chi(G) = f_G(0) - f_G(-1) =$

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[1] O. Knill, Dehn-Sommerville from Gauss-Bonnet, https://arxiv.org/abs/1905.04831, 2019

$\int_{-1}^{0} f'_G(x)\, dx$. Because the formula computes $f_G(x)$ recursively through unit spheres, allows to compute clique numbers of most graphs quickly. In general, it is a notoriously difficult NP complete problem.

2.4. Example: the icosahedron G has 12 vertices, 30 edges and 20 triangles. The polynomial is $f_G(x) = 1 + 12x + 30x^2 + 20x^3$. And the Euler characteristic is $\chi(G) = f(0) - f(-1) = 12 - 30 + 20 = 2$. René Descartes already noticed that Platonic solids lead to $\chi(G) = 2$ and wrote it down in an encrypted way in a secret notebook since mathematical theorems were at that time considered treasures. The unit sphere of a vertex is a circular graph with 5 vertices and 5 edges has the polynomial $f_{S(v)}(x) = 1 + 5x + 5x^2$. Summing over all 12 vertices gives $12 + 60x + 60x^2$. This agrees with $f'_G(x) = 12 + 60x + 60x^2$. Integrate this from -1 to 0 gives the already computed $\chi(G) = 2$.

2.5. Example: for the Echidnahedron with 92 vertices, 270 edges, 240 triangles and 60 tetrahedra, the polynomial is $f_G(x) = 1 + 92x + 270x^2 + 240x^3 + 60x^4$. The unit spheres are 60 triangles with $f_{S(v)}(x) = 1 + 3x + 3x^2 + x^3$ then 12 unit spheres with $f_{S(v)}(x) = 1 + 10x + 15x^2 + 5x^3$ and 20 unit spheres with $f_{S(v)}(x) = 1 + 12x + 18x^2 + 6x^3$. They all sum up to $92 + 540x + 720x^2 + 240x^3$ which agrees with $f'_G(x)$.

2.6. Problem 1: Repeat the computation for the octahedron G. So, compute $f_G(x)$, find the unit spheres $S(v)$ and compute $f_{S(v)}(x)$ the verify the formula from the theorem and compute the Euler characteristic.

2.7. Problem 2: Do the same for the **tetraxis hexahedron**: find $f_G(x)$ and $\chi(G)$. It is a cube on which we add a central node on each face and connect it to the 4 vertices of the square. So there are $8 + 6 = 14$ vertices and $6 * 4$ triangles.

2.8. The graphs of the polynomials are very interesting. One can study their roots, their maxima minima etc. When plotting the functions $f_G(x)$ for polynomials, we often notice some symmetry. This is called a **Dehn-Sommerville** property. Let us call a graph a **polyhedron** if all unit spheres $S(v)$ are circular graphs of 4 or more vertices.

2.9. Problem 3: Plot the graphs of $f_G(x)$ for the polyhedra: icosahedron, octahedron and tetraxis hexahedron.

Corollary: If G is a polyhedron, then $f_G(x)$ has a root at $x = -1/2$.

This follows from the theorem. All $g(x) = f_{S(v)}(x)$ have the property that $g(x + 1/2)$ even. Therefore the sum of such functions is even and so $f'_G(x + 1/2)$ is even. But that means that $f_G(x + 1/2)$ is odd, so that $x = -1/2$ is a root.

2.10. If G is a d-dimensional Dehn-Sommerville space define the function $h_G(x) = (x - 1)^{d+1} f_G(1/(x - 1))$. In our case, if triangles are the largest cliques, then $d = 2$ and $h_G(x) = (x - 1)^3 f_G(1/(x - 1))$. The **Dehn-Sommerville relations** assert that the h-vector is **palindromic**, meaning that $h_i = h_{d+1-i}$ for all $i = 0, \ldots, d + 1$. These identities is what the Dehn and Sommerville have noticed first. If G satisfies Dehn-Sommerville, then $f(-1) = (-1)^d f(0) = (-1)^d$ and so $\chi(G) = 1 - f(-1) = 1 + (-1)^d$. In the case $d = 2$, we get $\chi(G) = 2$. For the icosahedron, $f_G(x) = 1 + 12x + 30x^2 + 20x^3$,

we have $h_G(x) = 1 + 9x + 9x^2 + x^3$ and indeed $(1, 9, 9, 1)$ is a polindrome. **Problem 4:** Compute $h_G(x)$ for the octahedron.

2.11. The theorem we have seen is very general and holds for any network. **Problem 5:** Draw your own graph (chose something you can actually do), then build all functions $f_{S(v)}(x)$ and compare their sum with the derivative $f'_G(x)$.
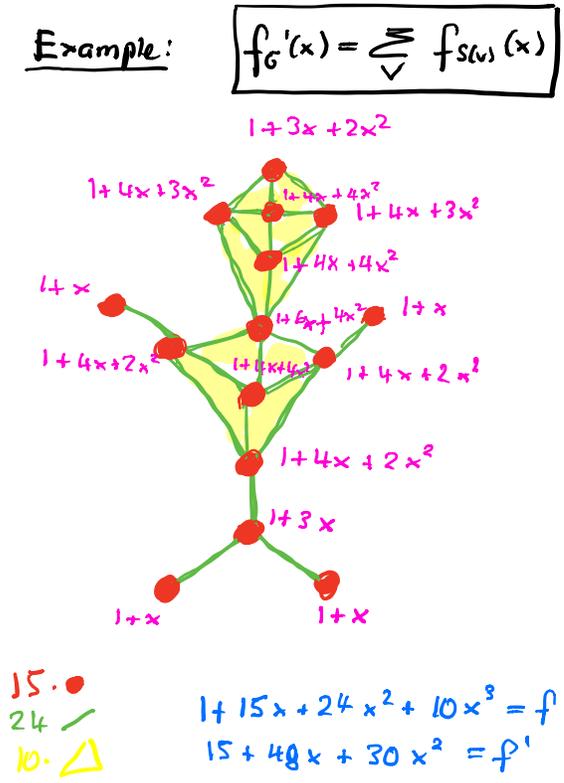

FIGURE 2. A hand-drawn graph. We compute at every vertex v the function $f_{S(v)}(x)$ and them up to get the derivative of $f_G(x)$. Pretty cool. The coefficients of the linear part produce the **Euler Handshake theorem** telling that 2 times the number of edges is the sum over all vertex degrees.