Unit 1: What is calculus?

1.1. Calculus deals with two themes: **taking differences** and **summing things up**. Differences measure how data **change**, sums quantify how quantities **accumulate**. The process of taking differences measures a **rate of change**. A limiting produced gives **the derivative**. The process of **summation** produces the **integral**. The two operations are related by the fundamental theorem of calculus. In this first lecture, we look at functions which are evaluated on the set integers and where there is no need for limits. It allows us to illustrate a major benefit of calculus: it gives us the ability to predict the future by analyzing the past.

1.2. Can you figure the next entry of the sequence of numbers

$$0, 3, 8, 15, 24, 35, 48, \ldots ?$$

When solving such a riddle, we already use already a basic idea of calculus. You might see that the differences

$$3, 5, 7, 9, 11, 13, \ldots$$

show a pattern. Taking differences again gives

$$2, 2, 2, 2, 2, 2, \ldots$$

Now, we can go back to the previous sequence and see that that the next term is 15. Looking at the original sequence gives $48 + 15 = 63$. Seeing such a difference pattern allows to get the future entries of the process. This observation is important.

1.3. Let us rewrite what we just did using the concept of a **function**. A function f takes an **input** x and gives an **output** called $y = f(x)$. The sequence we have just seen is then the function $f(1) = 0, f(2) = 3, f(3) = 8, f(4) = 15, f(5) = 24, \ldots$ Define now a new function Df by $Df(x) = f(x+1) - f(x)$. It is a rate of change which we also call a "derivative". Write also $f'(x)$ instead of $f(x)$. We have $f'(1) = 3 - 0 = 3, f'(2) = 8 - 3 = 5, f'(3) = 15 - 8 = 7, \ldots$. Now, we can take the derivative again and define $f''(n) = f'(n+1) - f'(n)$. The function f'' is the function where the derivative has been applied twice. We have seen $f''(1) = 2, f''(2) = 2, f''(3) = 2, \ldots$. The second derivative is constant. We have dealt with data which measure a constant acceleration.

1.4. Functions can be visualized graphically in the form of a **graphs** $y = f(x)$. To do so, we draw two perpendicular axes, the x-axis and the y-axis and mark down every pair $(x, f(x))$ in this **Euclidean plane**.
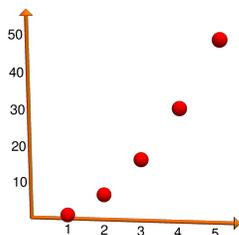
FIGURE 1.   When plotting the sequence of numbers in the coordinate plane, the function is visualized as a graph.

1.5. When the first mathematicians were recording numbers, they marked them into **tally sticks**. An artifact from tens of thousands of years ago is the **Ishango bone**. We can look at the marks as a constant function

$$1, 1, 1, 1, ... \ .$$

Over the next thousands of years, humans figured out to represent numbers as symbols like

$$1, 2, 3, 4, \ldots \ .$$

We see that $1 = 1, 2 = 1 + 1, 3 = 1 + 1 + 1$ etc. If we look at this counting function $f(x) = x$, it satisfies $f'(x) = 1$, the constant function and $f''(x) = 0$.

1.6. We can now ask which function g has the property that $g' = f$. The function g represents the summation of the terms. For example $f(5) = 0 + 1 + 2 + 3 + 4$ and $f(3) = 0 + 1 + 2 + 3$, then $f(4 + 1) - f(4) = 4$. We see that if we define $g = Sf$ as

$$Sf(x) = f(0) + f(1) + f(2) + \cdots + f(x - 1)$$

then $g(x + 1) - g(x) = f(x)$. Can we get a formula for the function g?

1.7. The new function g satisfies $g(1) = 1, g(2) = 3, g(3) = 6$, etc. These numbers are called **triangular numbers**. From the function g we can get f back by taking difference:

$$Dg(n) = g(n + 1) - g(n) = f(n) \ .$$

For example $Dg(5) = g(6) - g(5) = 15 - 10 = 5$. And indeed this is $f(5)$. Finding a formula for the sum $Sf(n)$ is not so easy if you have not seen it yet. We have to find the n'th term in the sequence which starts with

$$1, 3, 6, 10, 15, 21, \ldots$$

1.8. Legend tells that when **Karl-Friedrich Gauss** was a 9 year old school kid, his teacher, Mr. Büttner gave him the task to sum up the first 100 positive integers $1 + 2 + \cdots + 100$. Gauss did not want to do this tedious work and looked for a better way to do it. He discovered that pairing the numbers up would simplify the summation He would write the sum as $(1 + 100) + (2 + 99) + \cdots + (50 + 51)$ so that the answer is $g(x) = x(x - 1)/2 = 5050$. We have now an explicit expression for the sum function. Lets apply the difference function again: $Dg(x) = x(x + 1)/2 - x(x - 1)/2 = x = f(x)$.

1.9. Let us add up the new sequence again and compute $h = Sg$. We get the sequence $0, 1, 4, 10, 20, 35, \ldots$ called **tetrahedral numbers**. The reason is that one can use $h(n)$ balls to build a tetrahedron of side length n. For example, we need $h(4) = 20$ golf balls to build a tetrahedron of side length 4. The formula which holds for h is $\boxed{h(x) = x(x-1)(x-2)/6}$. In the worksheet we will check that summing the differences gives the function back.

1.10. The general relation
$$\boxed{SDf(x) = f(x) - f(0), \qquad DSf(x) = f(n)}$$
already is a version of the **fundamental theorem of calculus**. It will lead to the **integral** $\boxed{\int_0^x f(x) \ dx}$, **derivative** $\boxed{\frac{d}{dx} f(x)}$ and the **fundamental theorem of calculus**
$$\boxed{\int_0^x \tfrac{d}{dt} f(t) \ dt = f(x) - f(0), \qquad \tfrac{d}{dx} \int_0^x f(t) \ dt = f(x)}$$

1.11. This is a fantastic result. The goal of this course is to understand this theorem, and to apply it. Note that if we define $[n]^0 = 1, [n]^1 = n, [n]^2 = n(n-1)/2, [n]^3 = n(n-1)(n-2)/6$ then $D[n] = [1], D[n]^2 = 2[n], D[n]^3 = 3[n]^2$ and in general
$$\boxed{\tfrac{d}{dx}[x]^n = n[x]^{n-1}}$$

EXAMPLES

1.12. The **Fibonnacci sequence** $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ satisfies $f(x) = f(x-1) + f(x-2)$. It defines a function on the positive integers. For example, $f(6) = 8$. What is the function $g = Df$? We can assume $f(0) = 0$. **Solution:** Taking difference produces a shifted version $\boxed{Df(x) = f(x-1)}$.

1.13. Take the same function f given by the sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ but now compute the function $h(n) = Sf(n)$ obtained by summing the first n numbers up. It gives the sequence $1, 2, 4, 7, 12, 20, 33, \ldots$. What sequence is that?
Solution: Because $Df(x) = f(x-1)$ we have $f(x) - f(0) = SDf(x) = Sf(x-1)$ so that $Sf(x) = f(x+1) - f(1)$. Summing the Fibonnacci sequence produces the Fibonnacci sequence shifted to the left with $f(2) = 1$ subtracted.

1.14. Find the next term in the sequence 2 6 12 20 30 42 56 72 90 110 132 .
Solution: Take differences

$$
\begin{array}{ccccccccccc}
2 & 6 & 12 & 20 & 30 & 42 & 56 & 72 & 90 & 110 & 132 \\
2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{array} .
$$

Now we can add an additional number, starting from the bottom and working us up.

$$
\begin{array}{cccccccccccc}
2 & 6 & 12 & 20 & 30 & 42 & 56 & 72 & 90 & 110 & 132 & \boxed{156} \\
2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & \boxed{24} \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & \boxed{2}
\end{array}
$$

1.15. Look at the function $f(n)$ which gives the n'th prime number. Lets look at the derivatives $D^k f$ but take the absolute value $|D^k(f)|$. In other words, we study $T(f)(n) = |f(n+1) - f(n)|$. Let's see

n=	1	2	3	4	5	6	7	8	9	...
$f(n) =$	2	3	5	7	11	13	17	23	29	...
$Tf(n) =$	**1**	2	2	4	2	4	2	4	6	...
$T^2 f(n) =$	**1**	0	2	2	2	2	2	2	4	...
$T^3 f(n) =$	**1**	2	0	0	0	0	0	2	0	...

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HOMEWORK

Problem 1.1: Predict the future and find the next term in the sequence
$$2, 10, 30, 68, 130, 222, 350, 520, 738, 1010, 1342, \ldots.$$

Problem 1.2: The sequence of odd numbers $f(x) = 2x + 1$ starts with $1, 3, 5, 7, 9, 11, 13, \ldots$. We want to find a function $g(x)$ which has the property $Dg(x) = f(x)$. To do so, play around and compute $Sf(1) = 1, Sf(2) = 1+3, Sf(3) = 1+3+5$ etc until you see a pattern. Now guess a formula for $g(x) = Sf(x) = f(0) + f(1) + f(2) + \cdots + f(x-1)$ and verify algebraically that $Dg(x) = g(x+1) - g(x) = f(x)$.

Problem 1.3: The function $f(x) = 2^x$ is called the **exponential function**. We have for example $f(0) = 1, f(1) = 2, f(2) = 4, \ldots$. Verify that this function satisfies the equation $\boxed{Df(x) = f(x)}$.

Problem 1.4: Lets draw a polygon with n sides. If we look at possible combinations of vertices which are connected to each other, we get a function $f(n)$. For example, for a pentagon, there are 5 single points, 5 edges so that $f(5) = 10$. For the hexagon, there are 6 vertices, 9 edges and 2 triangles summing up to $f(6) = 17$. Here are the 10 first data points. Find a rule for the next one and compute the next 3 entries.

$n =$	1	2	3	4	5	6	7	8	9	10
$f(n) =$	1	2	3	6	10	17	28	46	75	122

Problem 1.5: Check in each of the following two cases that $g = Sf$ holds. You can do that by verifying that $Dg = f$.
a) For $f(x) = x + 1$ we have $g(x) = x(x+1)/2$.
b) For $f(x) = x(x-1)/2$ we have $g(x) = x(x-1)(x-2)/6$.

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[1]The **Gilbreath conjecture** of 1959 claims that the first entries all reach and remain 1 when we continue. This is still an open mathematical problem.