INTRODUCTION TO CALCULUS

MATH 1A

Unit 25: Integration by parts

25.1. Integrating the product rule $(uv)' = u'v + uv'$ gives the method **integration by parts**. It complements the method of substitution we have seen last time. As a rule of thumb, always try first to **1) simplify a function and integrate using known functions**, then **2) try substitution** and finally **3) try integration by parts**.

$$\int \text{u(x)}\ \text{v'}\ (x)dx = u(x)v(x) - \int u'(x)v(x)\ dx.$$

Example: To see how integration by parts work, lets try to find $\int x\sin(x)\ dx$. First identify what you want to differentiate and call it u, the part to integrate is called v'. Now, write down uv and subtract a new integral which integrates $u'v$:

$$\int \text{x}\ \sin(x)\ dx = \text{x}\ (-\cos(x)) - \int 1\ (-\cos(x)\ )\ dx = -x\cos(x) + \sin(x) + C\ dx\ .$$

In class, I will will stream line this by just placing an arrow down under the expression you differentiate and an arrow up under the expression you integrate. You remember to first integrate, then subtract the integral of the expression where you both integrate and differentiate. If you like to write down the u, v, do so and remember $\boxed{\int udv = uv - \int vdu}$.

Example: Find $\int xe^x\ dx$. **Solution.** You want to differentiate x and integrate e^x.

$$\int \text{x}\ \exp(x)\ dx = x\exp(x) - \int 1\cdot\exp(x)\ dx = x\exp(x) - \exp(x) + C\ dx\ .$$

Example: Find $\int \log(x)\ dx$. **Solution.** While there is only one function here, we need two to use the method. Let us look at $\log(x)\cdot 1$:

$$\int \log(x)\ 1\ dx = \log(x)x - \int \frac{1}{x}x\ dx = x\log(x) - x + C\ .$$

Example: Find $\int x\log(x)\ dx$. **Solution.** Since we know from the previous problem how to integrate log we could proceed by taking $x = u$. We can also take $u = \log(x)$ and $dv = x$:

$$\int \log(x)\ \text{x}\ dx = \log(x)\frac{x^2}{2} - \int \frac{1}{x}\frac{x^2}{2}\ dx$$

which is $\log(x)x^2/2 - x^2/4$.

25.2. We see that it is often better to differentiate log first. The word LIATE explained below tells which functions we want to call u and differentiate.

Example: **Marry go round:** Find $I = \int \sin(x)\exp(x)\ dx$. **Solution.** Lets integrate $\exp(x)$ and differentiate $\sin(x)$.

$$= \sin(x)\exp(x) - \int \cos(x)\exp(x)\ dx\ .$$

Lets do it again:

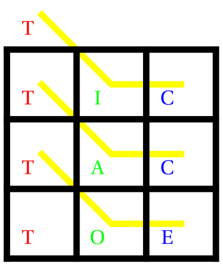
$$= \sin(x)\exp(x) - \cos(x)\exp(x) - \int \sin(x)\exp(x)\ dx.$$

We moved in circles and are stuck! But wait. Are we really? We have now derived an identity

$$I = \sin(x)\exp(x) - \cos(x)\exp(x) - I$$

which we can solve for I and get $\boxed{I = [\sin(x)\exp(x) - \cos(x)\exp(x)]/2.}$

Tic-Tac-Toe



Integration by parts can become complicated if it has to be done several times. Keeping the order of the signs can be especially daunting. Fortunately, there is a powerful **tabular integration by parts method**. It has been called **"Tic-Tac-Toe"** in the movie Stand and deliver. Lets call it **Tic-Tac-Toe** therefore.

Example: Find the anti-derivative of $(x-1)^3 e^{2x}$. **Solution:**

$(x-1)^3$	$\exp(2x)$	
$3(x-1)^2$	$\exp(2x)/2$	$\oplus$
$6(x-1)$	$\exp(2x)/4$	$\ominus$
6	$\exp(2x)/8$	$\oplus$
0	$\exp(2x)/16$	$\ominus$

The anti-derivative is

$$(x-1)^3 e^{2x}/2 - 3(x-1)^2 e^{2x}/4 + 6(x-1)e^{2x}/8 - 6e^{2x}/16 + C\ .$$

Example: Find the anti-derivative of $x^2\cos(x)$. **Solution:**

x^2	$\cos(x)$	
$2x$	$\sin(x)$	$\oplus$
2	$-\cos(x)$	$\ominus$
0	$-\sin(x)$	$\oplus$

The anti-derivative is $x^2 \sin(x) + 2x\cos(x) - 2\sin(x) + C$.

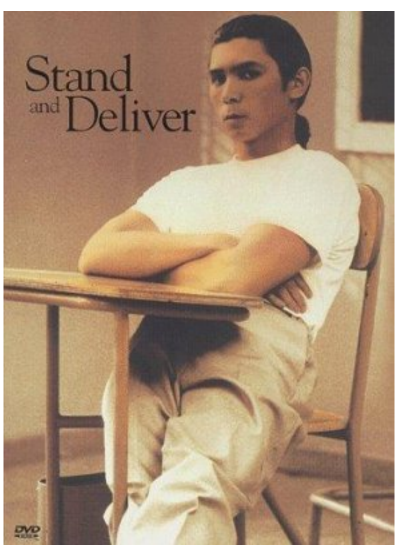
Example: More extreme Find the anti-derivative of $x^7 \cos(x)$. **Solution:**

x^7	$\cos(x)$	
$7x^6$	$\sin(x)$	$\oplus$
$42x^5$	$-\cos(x)$	$\ominus$
$120x^4$	$-\sin(x)$	$\oplus$
$840x^3$	$\cos(x)$	$\ominus$
$2520x^2$	$\sin(x)$	$\oplus$
$5040x$	$-\cos(x)$	$\ominus$
5040	$-\sin(x)$	$\oplus$
0	$\cos(x)$	$\ominus$

The anti-derivative is

$$\begin{aligned} F(x) &= x^7\sin(x) + 7x^6\cos(x) - 42x^5\sin(x) \\ &- 210x^4\cos(x) + 840x^3\sin(x) + 2520x^2\cos(x) \\ &- 5040x\sin(x) - 5040\cos(x) + C\ . \end{aligned}$$

25.3. Do this without this method and you see the value of the method.
[1] [2] [3]:



I myself learned the method from the movie "Stand and Deliver", where **Jaime Escalante** of the Garfield High School in LA uses the method. It can be traced down to an article of V.N. Murty. The method realizes in a clever way an iterated integration by parts method:

$$\begin{aligned} \int fg\,dx &= fg^{(-1)} - f^{(1)}g^{-2} + f^{(2)}g^{(-3)} - \dots \\ &- (-1)^n \int f^{(n+1)}g^{(-n-1)}\,dx \end{aligned}$$

The method can be verified by induction because the f function is differentiated again and again and the g function is integrated again and again. The alternating minus-plus-signs come from the fact that we subtract again an integral. We always pair a k'th derivative with a $k+1$'th integral and take the sign $(-1)^k$.

Coffee or Tea?

[1] V.N. Murty, Integration by parts, Two-Year College Mathematics Journal 11, 1980, p. 90-94.
[2] D. Horowitz, Tabular Integration by Parts, College Mathematics Journal, 21, 1990, p. 307-311.
[3] K.W. Folley, integration by parts, American Mathematical Monthly 54, 1947, p. 542-543

25.4. When doing integration by parts, We want to try first to differentiate **L**ogs, **I**nverse trig functions, **P**owers, **T**rig functions and **E**xponentials. This can be remembered as **LIPTE** which is close to "lipton" (the tea).

For coffee lovers, there is an equivalent one: **L**ogs, **I**nverse trig functions, **A**lgebraic functions, **T**rig functions and **E**xponentials which can be remembered as **LIATE** which is close to "latte" (the coffee).

Whether you prefer to remember it as a "coffee latte" or a "lipton tea" is up to you.

There is even a better method, the "method of the opportunist":

Just integrate what you can integrate and differentiate the rest.

An don't forget to consider integrating 1, if nothing else works.

Homework

It is the season of holidays worldwide: Passover, Easter, Qingming, Holi, Ramadan, we celebrate wellness day! No homework therefore. We will in Lecture 26 add two or three problems from the following.

Problem 25.1: Integrate $\int x^3 \log(x)\, dx$.

Problem 25.2: Integrate $\int x^5 \sin(x)\, dx$

Problem 25.3: Find the anti derivative of $\int 2x^6 \exp(x)\, dx$.

Problem 25.4: Find the anti derivative of $\int \sqrt{x} \log(x)\, dx$.

Problem 25.5: Find the anti derivative of $\int \sin(x) \exp(-x)\, dx$.

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