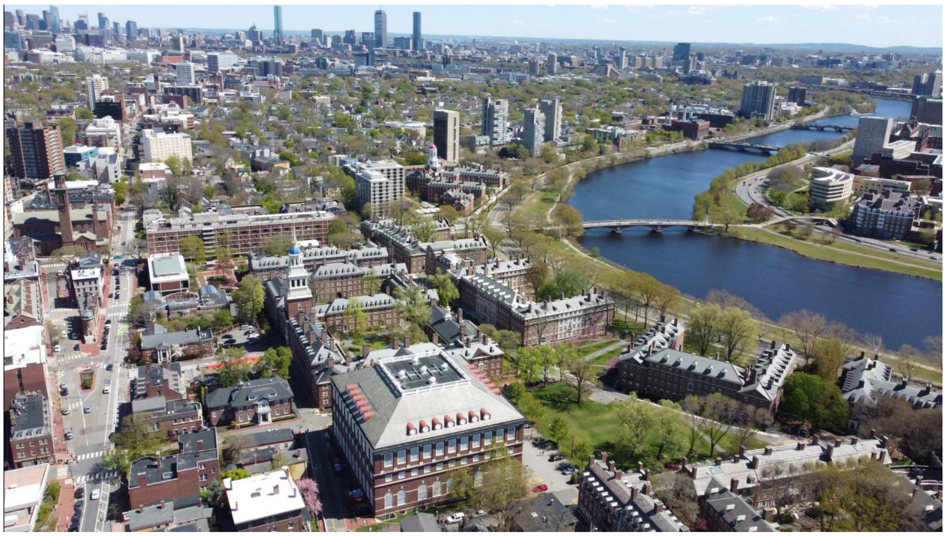
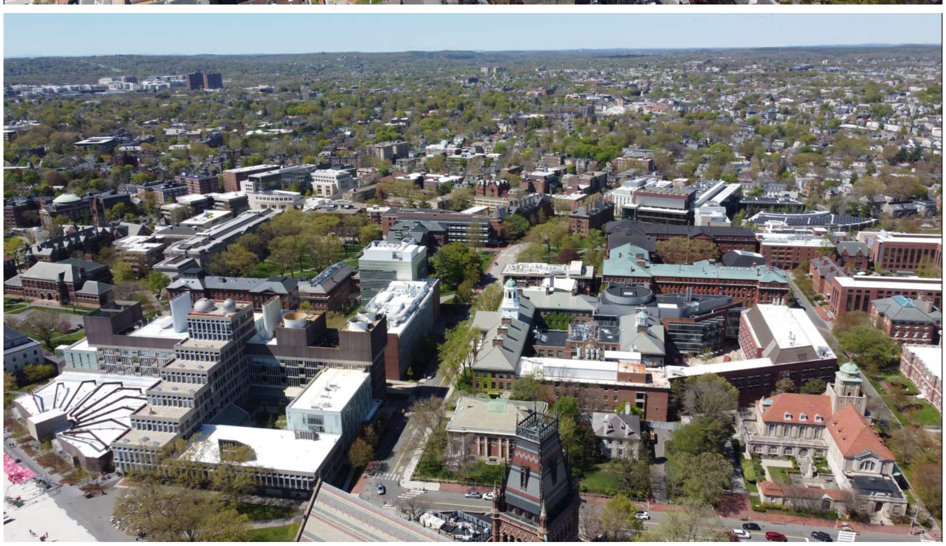
INTRODUCTION TO CALCULUS

OLIVER KNILL





Harvard University, aerial pictures from May 1, 2021.

Oliver Knill, knill@math.harvard.edu, Math 1a, Harvard College, Spring 2021

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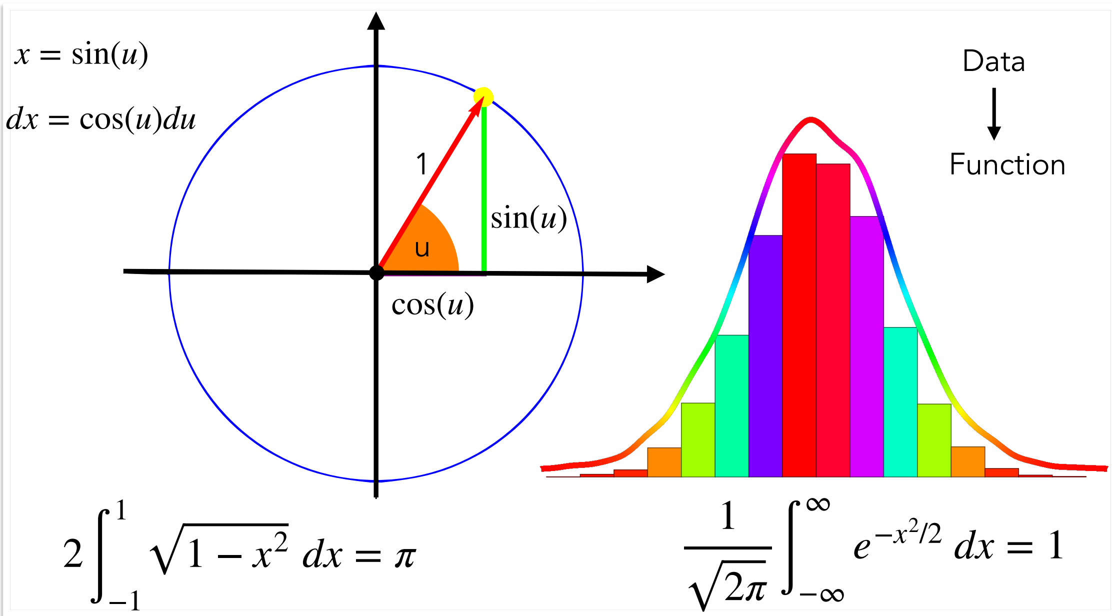

Content: Pages 1-149 contain 36 lectures with homework. Pages 148-156 are four data projects, pages 157-313 exam collection.

Introduction

0.1. This course is an introduction to calculus, taught during the years 2011-2014 and 2020-2021 at the Harvard college. While no previous calculus exposure is expected, basic pre-calculus skills like geometry and algebra are assumed. Even if you have seen some calculus before, a college single variable calculus course like this one will lead to a deeper, more **conceptual understanding** of the subject. It allows to see the beauty and elegance of a mathematical theory and appreciate its **applications**. While both concepts and applications are very important, also the mastery of **skills** is pivotal. Especially when doing the first steps in a new field, one has to focus on skills. Fortunately, procedures are faster to learn and teach than **insight** which requires more time as it requires **experience** and the ability to connect the dots and see similarities, patterns and being able to ask questions.

0.2. A good strategy is not to worry at first too much about reaching the ultimate picture but to focus on mastering isolated skills. The unfortunate dogma "you need first to understand!" can lead to learning blocks. Whenever one embarks into a new area of knowledge or activity, one always faces a similar challenge: at first one has to get acquainted with language and jargon, then one learns how to work out things and finally one sees patterns and gains insight and can build bridges between different already grown knowledge patches. It is important at first to just enjoy the learning while doing it. The understanding will come naturally. This is the same for all learning: if you learn to play a music instrument, a new language, to cook or to climb mountains, you first want to know the notes, the words, the ingredients and the gear. Then you learn how to play, to speak, to follow recipes or climb following a mountain guide before you start to improvise, write your own text, to create new meals or discover uncharted climbing routes.

0.3. You might wonder why it is necessary to learn a single variable theory. Isn't the world vastly **multi-dimensional**? It turns out that the **one-dimensional** point of view, the development of a single quantity over time, is extremely important. If you study the motion of the universe for example, you are interested in its expansion rate, which is a function of time and so subject to single variable calculus. If you study the spread of a virus, you are interested in the number of infected as a function of time. If you study climate or weather, you can be interested in the average global temperature over time, like decades or centuries. These are **functions of one variable**, despite the fact that the underlying mechanisms are complex systems like partial differential equations. If you are interested in finance, you might be concerned in the stock prize of a single company over time. If you probe a probability space with a random variable, then you are interested in the distribution. These are all functions of one variable again. Time is one dimensional. Single variable calculus therefore is a window to some of the deepest secrets of knowledge.

0.4. Calculus also is a large part of our scientific cultural heritage. Knowing about the historical development helps us to see today, where the difficulties are. A student of calculus essentially faces the same challenges than our ancestors who developed the field. History also produces story lines and add a dramatic element. It is only a dozen generations ago when humanity did not understood the notion of limit. In antiquity, the subject had been part of sometimes controversial philosophy. Paradoxa formulated already in antiquity illustrate the confusion which existed then. Today we have a crystal clear picture. While the ideas of calculus already trace back to the time of ancient Greece, in particular to the time of Archimedes, the subject has exploded into a **powerful tool** during last few centuries. It is now a pivotal **theoretical foundation** for other mathematical areas and scientific fields. Without exaggeration it is safe to say that calculus is one of most amazing scientific and cultural achievements of humanity.

0.5. Calculus consists of differential and integral calculus. Differential calculus studies "change", integral calculus deals with "accumulation". The fundamental theorem of calculus links the two. The subject is very applicable to problems from other scientific disciplines. Calculus is not only important because of its content and applications like life sciences (example: tomography), data science (example: compute correlations), internet (example: networks), artificial intelligence (example: machine learning), geography (example: data visualization), movie and game industry (computer graphics), the ideas of calculus also enter in disguised form, in statistics, economics, computer science, in art or music theory. Do not forget that we primarily want to learn the nuts and bolts and down-to-earth techniques.

0.6. The use of computers and computer algebra systems or online tools to experiment with the mathematical structures is encouraged. The use of laptops or tablets in class to take notes is of course perfect. No computer, phone or tablet of any type will be permitted however during exams except possibly as a writing tool, when tests are taken remotely. If you get computer assistance for homework, acknowledge it in the homework. We do recommend that you work out most of the work on paper (or electronic paper). The material sticks better when you write mathematical formulas and procedures by hand. It enhances long term retention and prepares you for exams.

0.7. We do not follow any book or previous course. There are many good books. A popular text which has proven to be useful in the last decades is "Single Variable Calculus: Concepts and Contexts", by James Stewart. Our homework problems are of similar difficulty than in such textbooks; some of them are trickier and intended to trigger discussion with other students or teaching staff. I recommend to attack each homework problem first on your own. This helps you to develop independent thinking and **problem solving skills** and prepare for the exams. You should get comfortable with the situation of being stuck at first. But even if you can solve a problem, it is helpful to discuss them with others. Mathematics is also a social activity.

Cropped from "School of Athens" Fresco by Raphael (a picture in the public domain)

Oliver Knill, January 10th- May 10th, 2021.

Oliver Knill, knill@math.harvard.edu, Math 1a, Harvard College, Spring 2021

INTRODUCTION TO CALCULUS

MATH 1A

Unit 1: What is calculus?

Lecture

1.1. Calculus deals with two themes: **taking differences** and **summing things up**. Differences measure how data **change**, sums quantify how quantities **accumulate**. The process of taking differences measures a **rate of change**. A limiting produced gives **the derivative**. The process of **summation** produces the **integral**. The two operations are related by the fundamental theorem of calculus. In this first lecture, we look at functions which are evaluated on the set integers and where there is no need for limits. It allows us to illustrate a major benefit of calculus: it gives us the ability to predict the future by analyzing the past.

1.2. Can you figure the next entry of the sequence of numbers

$$0, 3, 8, 15, 24, 35, 48, \dots ?$$

When solving such a riddle, we already use already a basic idea of calculus. You might see that the differences

$$3, 5, 7, 9, 11, 13, \dots$$

show a pattern. Taking differences again gives

$$2, 2, 2, 2, 2, 2, \dots$$

Now, we can go back to the previous sequence and see that that the next term is 15. Looking at the original sequence gives $48 + 15 = 63$. Seeing such a difference pattern allows to get the future entries of the process. This observation is important.

1.3. Let us rewrite what we just did using the concept of a **function**. A function f takes an **input** x and gives an **output** called $y = f(x)$. The sequence we have just seen is then the function $f(1) = 0, f(2) = 3, f(3) = 8, f(4) = 15, f(5) = 24, \dots$ Define now a new function Df by $Df(x) = f(x+1) - f(x)$. It is a rate of change which we also call a "derivative". Write also $f'(x)$ instead of $f(x)$. We have $f'(1) = 3 - 0 = 3, f'(2) = 8 - 3 = 5, f'(3) = 15 - 8 = 7, \dots$. Now, we can take the derivative again and define $f''(n) = f'(n+1) - f'(n)$. The function f'' is the function where the derivative has been applied twice. We have seen $f''(1) = 2, f''(2) = 2, f''(3) = 2, \dots$. The second derivative is constant. We have dealt with data which measure a constant acceleration.

1.4. Functions can be visualized graphically in the form of a **graphs** $y = f(x)$. To do so, we draw two perpendicular axes, the x-axis and the y-axis and mark down every pair $(x, f(x))$ in this **Euclidean plane**.

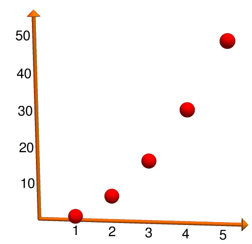

FIGURE 1. When plotting the sequence of numbers in the coordinate plane, the function is visualized as a graph.

1.5. When the first mathematicians were recording numbers, they marked them into **tally sticks**. An artifact from tens of thousands of years ago is the **Ishango bone**. We can look at the marks as a constant function

$$1, 1, 1, 1, ...$$

Over the next thousands of years, humans figured out to represent numbers as symbols like

$$1, 2, 3, 4, \dots .$$

We see that $1 = 1, 2 = 1 + 1, 3 = 1 + 1 + 1$ etc. If we look at this counting function $f(x) = x$, it satisfies $f'(x) = 1$, the constant function and $f''(x) = 0$.

1.6. We can now ask which function g has the property that $g' = f$. The function g represents the summation of the terms. For example $f(5) = 0 + 1 + 2 + 3 + 4$ and $f(3) = 0 + 1 + 2 + 3$, then $f(4 + 1) - f(4) = 4$. We see that if we define $g = Sf$ as

$$Sf(x) = f(0) + f(1) + f(2) + \cdots + f(x - 1)$$

then $g(x + 1) - g(x) = f(x)$. Can we get a formula for the function g?

1.7. The new function g satisfies $g(1) = 1, g(2) = 3, g(3) = 6$, etc. These numbers are called **triangular numbers**. From the function g we can get f back by taking difference:

$$Dg(n) = g(n + 1) - g(n) = f(n) .$$

For example $Dg(5) = g(6) - g(5) = 15 - 10 = 5$. And indeed this is $f(5)$. Finding a formula for the sum $Sf(n)$ is not so easy if you have not seen it yet. We have to find the n'th term in the sequence which starts with

$$1, 3, 6, 10, 15, 21, \dots$$

1.8. Legend tells that when **Karl-Friedrich Gauss** was a 9 year old school kid, his teacher, Mr. Büttner gave him the task to sum up the first 100 positive integers $1 + 2 + \cdots + 100$. Gauss did not want to do this tedious work and looked for a better way to do it. He discovered that pairing the numbers up would simplify the summation He would write the sum as $(1 + 100) + (2 + 99) + \cdots + (50 + 51)$ so that the answer is $g(x) = x(x - 1)/2 = 5050$. We have now an explicit expression for the sum function. Lets apply the difference function again: $Dg(x) = x(x+1)/2 - x(x-1)/2 = x = f(x)$.

1.9. Let us add up the new sequence again and compute $h = Sg$. We get the sequence $0, 1, 4, 10, 20, 35, \dots$ called **tetrahedral numbers**. The reason is that one can use $h(n)$ balls to build a tetrahedron of side length n. For example, we need $h(4) = 20$ golf balls to build a tetrahedron of side length 4. The formula which holds for h is $\boxed{h(x) = x(x-1)(x-2)/6}$. In the worksheet we will check that summing the differences gives the function back.

1.10. The general relation

$$\boxed{SDf(x) = f(x) - f(0), \qquad DSf(x) = f(n)}$$

already is a version of the **fundamental theorem of calculus**. It will lead to the **integral** $\boxed{\int_0^x f(x)\,dx}$, **derivative** $\boxed{\frac{d}{dx} f(x)}$ and the **fundamental theorem of calculus**

$$\boxed{\int_0^x \frac{d}{dt} f(t)\,dt = f(x) - f(0), \qquad \frac{d}{dx}\int_0^x f(t)\,dt = f(x)}$$

1.11. This is a fantastic result. The goal of this course is to understand this theorem, and to apply it. Note that if we define $[n]^0 = 1, [n]^1 = n, [n]^2 = n(n-1)/2, [n]^3 = n(n-1)(n-2)/6$ then $D[n] = [1], D[n]^2 = 2[n], D[n]^3 = 3[n]^2$ and in general

$$\boxed{\frac{d}{dx}[x]^n = n[x]^{n-1}}$$

Examples

1.12. The **Fibonnacci sequence** $1, 1, 2, 3, 5, 8, 13, 21, \dots$ satisfies $f(x) = f(x-1) + f(x-2)$. It defines a function on the positive integers. For example, $f(6) = 8$. What is the function $g = Df$? We can assume $f(0) = 0$. **Solution:** Taking difference produces a shifted version $\boxed{Df(x) = f(x-1)}$.

1.13. Take the same function f given by the sequence $1, 1, 2, 3, 5, 8, 13, 21, \dots$ but now compute the function $h(n) = Sf(n)$ obtained by summing the first n numbers up. It gives the sequence $1, 2, 4, 7, 12, 20, 33, \dots$. What sequence is that?
Solution: Because $Df(x) = f(x-1)$ we have $f(x) - f(0) = SDf(x) = Sf(x-1)$ so that $Sf(x) = f(x+1) - f(1)$. Summing the Fibonnacci sequence produces the Fibonnacci sequence shifted to the left with $f(2) = 1$ subtracted.

1.14. Find the next term in the sequence 2 6 12 20 30 42 56 72 90 110 132 .
Solution: Take differences

2	6	12	20	30	42	56	72	90	110	132
2	4	6	8	10	12	14	16	18	20	22
2	2	2	2	2	2	2	2	2	2	2

Now we can add an additional number, starting from the bottom and working us up.

2	6	12	20	30	42	56	72	90	110	132	**156**
2	4	6	8	10	12	14	16	18	20	22	**24**
2	2	2	2	2	2	2	2	2	2	2	**2**

1.15. Look at the function $f(n)$ which gives the n'th prime number. Lets look at the derivatives $D^k f$ but take the absolute value $|D^k(f)|$. In other words, we study $T(f)(n) = |f(n+1) - f(n)|$. Let's see

n=	1	2	3	4	5	6	7	8	9	...
$f(n) =$	2	3	5	7	11	13	17	23	29	...
$Tf(n) =$	$\boxed{1}$	2	2	4	2	4	2	4	6	...
$T^2f(n) =$	$\boxed{1}$	0	2	2	2	2	2	2	4	...
$T^3f(n) =$	$\boxed{1}$	2	0	0	0	0	0	2	0	...

[1]

Homework

Problem 1.1: Predict the future and find the next term in the sequence

$$2, 10, 30, 68, 130, 222, 350, 520, 738, 1010, 1342, \dots.$$

Problem 1.2: The sequence of odd numbers $f(x) = 2x + 1$ starts with $1, 3, 5, 7, 9, 11, 13, \dots$. We want to find a function $g(x)$ which has the property $Dg(x) = f(x)$. To do so, play around and compute $Sf(1) = 1, Sf(2) = 1+3, Sf(3) = 1+3+5$ etc until you see a pattern. Now guess a formula for $g(x) = Sf(x) = f(0) + f(1) + f(2) + \dots + f(x-1)$ and verify algebraically that $Dg(x) = g(x+1) - g(x) = f(x)$.

Problem 1.3: The function $f(x) = 2^x$ is called the **exponential function**. We have for example $f(0) = 1, f(1) = 2, f(2) = 4, \dots$. Verify that this function satisfies the equation $\boxed{Df(x) = f(x)}$.

Problem 1.4: Lets draw a polygon with n sides. If we look at possible combinations of vertices which are connected to each other, we get a function $f(n)$. For example, for a pentagon, there are 5 single points, 5 edges so that $f(5) = 10$. For the hexagon, there are 6 vertices, 9 edges and 2 triangles summing up to $f(6) = 17$. Here are the 10 first data points. Find a rule for the next one and compute the next 3 entries.

$n =$	1	2	3	4	5	6	7	8	9	10
$f(n) =$	1	2	3	6	10	17	28	46	75	122

Problem 1.5: Check in each of the following two cases that $g = Sf$ holds. You can do that by verifying that $Dg = f$.
a) For $f(x) = x + 1$ we have $g(x) = x(x+1)/2$.
b) For $f(x) = x(x-1)/2$ we have $g(x) = x(x-1)(x-2)/6$.

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[1] The **Gilbreath conjecture** of 1959 claims that the first entries all reach and remain 1 when we continue. This is still an open mathematical problem.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 2: Functions

Lecture

2.1. A **function** is a rule which assigns to a real number a new real number. The function $f(x) = x^3 - 2x$ for example assigns to the number $x = 2$ the value $2^3 - 4 = 4$. A function is assigned a **domain** A, the points where f is defined and a **codomain** B a set of numbers in which f is mapped to. The **range** is $f(A)$.

2.2. Many functions like $f(x) = x^2 - 2x$ are defined everywhere. In general, we assume that the domain is the place where the function is defined and the codomain is the set of real numbers and the range the set of numbers which are reached by f. Functions can also be defined on domains which are discrete. The prime function $p(n)$ which gives the n'th prime is also a function. The domain is $\mathbb{N} = \{1, 2, 3, \dots\}$ of natural numbers, the range B is the set of primes.

2.3. A function $g(x) = 1/x$ for example can not be evaluated at 0 so that the domain must exclude the point 0. Its range is also $\mathbb{R} \setminus \{0\}$, the set of real numbers without 0. The **inverse** of a function f is a function g such that $g(f(x)) = x$. The function $g(x) = \sqrt{x}$ for example is the inverse of the function $f(x) = x^2$ on its domain $\mathbb{R}^+ = [0, \infty)$. The function $f(x) = 1/x$ is its own inverse.

2.4. Here are a few examples. We will look at many of them in more detail during the lecture. Very important are polynomials, trigonometric functions, the exponential and the logarithmic function. Below we see some functions. The compound interest function can also be interpreted as an exponential. It will for $h \to 0$ go over to the exponential function. The logarithmic function as the inverse of the exponential function is only defined on the positive real axes.

constant	1	power	2^x
identity	x	exponential	$e^x = \exp(x)$
linear	$3x + 1$	logarithm	$\log(x) = \ln(x)$
quadratic	x^2	absolute value	$\lvert x \rvert$
cosine	$\cos(x)$	devil comb	$\sin(1/x)$
sine	$\sin(x)$	bell function	e^{-x^2}
compound interest	$\exp_h(x) = (1 + h)^{x/h}$	Agnesi	$\frac{1}{1+x^2}$
logarithms	$\log(x) = \ln(x)$	sinc	$\sin(x)/x$

We can build new functions by:

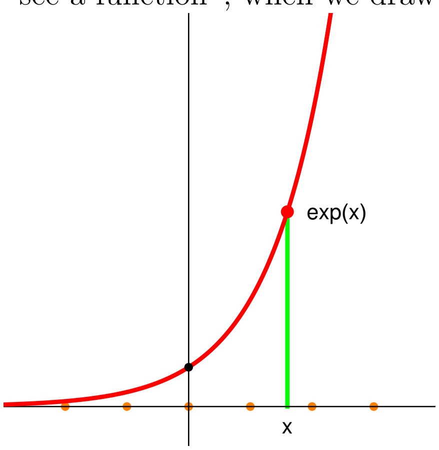
addition	$f(x) + g(x)$
multiplication	$f(x) * g(x)$
division	$f(x)/g(x)$
scaling	$2f(x)$
translation	$f(x+1)$
composing	$f(g(x))$
inverting	$f^{-1}(x)$

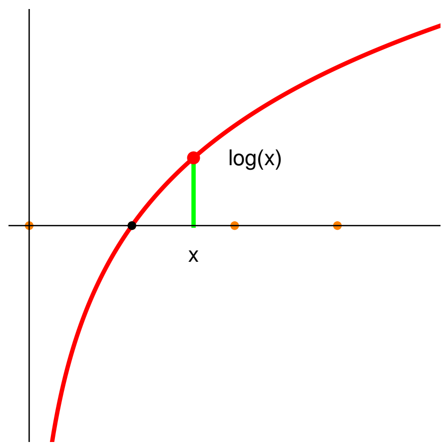
Important functions:

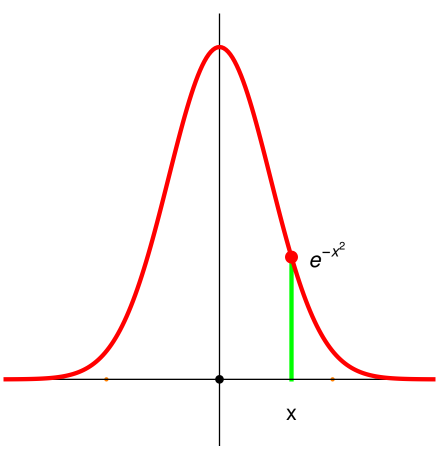
polynomials	$x^2 + 3x + 5$
rational functions	$(x+1)/(x^4+1)$
exponential	e^x
logarithm	$\log(x)$
trig functions	$\sin(x)$, $\tan(x)$
inverse trig functions	$\arcsin(x), \arctan(x)$.
roots	$\sqrt{x}, x^{1/3}$

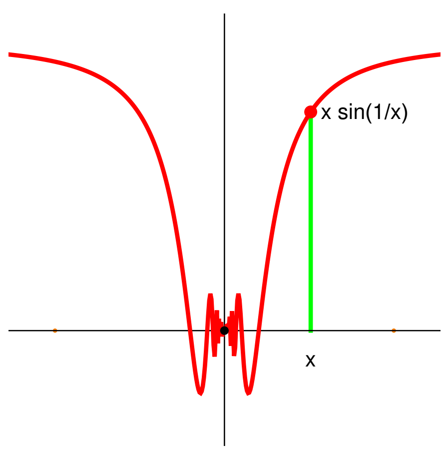
2.5. We will look at these functions **a lot** during this course. The logarithm, exponential and trigonometric functions are especially important. For some functions, we need to restrict the domain, where the function is defined. For the square root function $\sqrt{x}$ or the logarithm $\log(x)$ for example, we have to assume that the number x on which we evaluate the function is positive. We write that the domain is $(0, \infty) = \mathbf{R}^+$. For the function $f(x) = 1/x$, we have to assume that x is different from zero. Keep these three examples in mind.

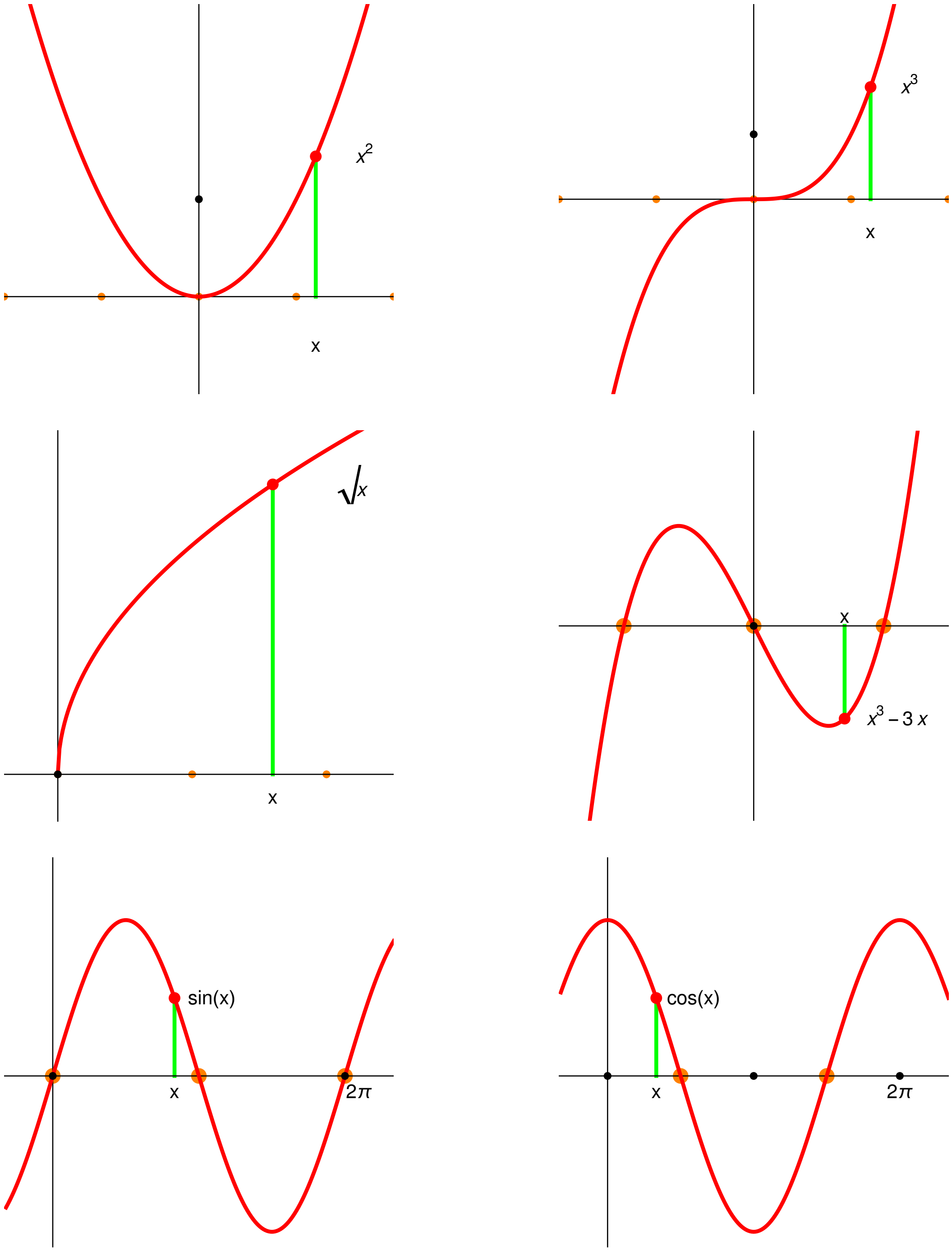
2.6. The **graph** of a function is the set of points $\{(x, y) = (x, f(x))\ \}$ in the plane, where x runs over the domain A of f. Graphs allow us to **visualize** functions. We can "see a function", when we draw the graph.











2.7. Definition: A function $f : A \to B$ is **invertible** if there is an other function g such that $g(f(x)) = x$ for all x in A and $f(g(y)) = y$ for all $y \in B$. The function g is the **inverse** of f. Example: $g(x) = \sqrt{x}$ is the inverse of $f(x) = x^2$ as a function from $A = [0, \infty)$ to $B = [0, \infty)$. You can check with the **horizontal line test** whether an inverse exists: draw the box with base A and side B, then every horizontal line should intersect the graph exactly once.

Homework

Here is the homework for this section.

Problem 2.1: Draw the function $f(x) = e^{x^2}\sin(4x)$ on the interval $[-5, 5]$. Its graph goes through the origin $(0, 0)$. You can use technology.
a) A function is called **odd** if $f(-x) = -f(x)$. Is f odd?
b) A function is called **even** if $f(-x) = f(x)$. Is f even?
c) What happens in general if a function f is both even and odd?

Problem 2.2: Determine from the following functions whether they are invertible. and write down the inverse if they are
a) $f(x) = x^{11} - 22$ from $A = \mathbb{R}$ to $B = \mathbb{R}$
b) $f(x) = \cos(x^5)$ from $A = [0, \pi/2]$ to $B = [0, 1]$
c) $f(x) = \sin(x)$ from $A = [0, \pi]$ to $B = [0, 1]$
d) $f(x) = \tan(x)$ from $A = (-\pi/2, \pi/2)$ to $B = \mathbb{R}$.
e) $f(x) = 1/(1 + x^2)$ from $A = [0, \infty)$ to $B = (0, 1]$.

Problem 2.3: a) Draw the graphs of $\exp_1(x) = 2^x, \exp_{1/4}(x) = (1 + 1/4)^{4x}$ and $\exp(x)$.
b) Draw the graphs the inverse of these functions.
You can use technology (like Desmos or Wolfram alpha) for a). For b), just "flip the graph" at the line $x = y$.

Problem 2.4: Try to plot the function $\exp(\exp(\exp(\exp(x))))$ on $[0, 1]$. You will see a sharp increase of the function after which the computer refuses to plot. Where is this value?

Problem 2.5: A function $f(x)$ has a **root** at $x = a$ if $f(a) = 0$. Find at least one root for each of the following functions.
a) $f(x) = x^7 - x^5$
b) $f(x) = \cos(x)$
c) $f(x) = 4\exp(-x^4)$
d) $f(x) = \log(x) = \ln(x)$
e) $f(x) = \sin(x) - 1$
f) $f(x) = \sec(x) = 1/\cos(x)$

(*) Here is how you to plot a function with Wolfram alpha:

```
http://www.wolframalpha.com/input/?i=Plot+sin(x)
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INTRODUCTION TO CALCULUS

MATH 1A

Unit 3: Limits

Lecture

3.1. The function $1/x$ is not defined everywhere. It blows up at $x = 0$ where we divide by zero. Sometimes however, a function can be **healed** at a point where it is not defined. A silly example is $f(x) = x^2/x$ which is initially not defined at $x = 0$ because we divide by x. The function can be "saved" by noticing that $f(x) = x$ for all x different from 0. Functions often can be continued to "forbidden" places if we write the function differently. This can involves dividing out a common factor. Here are some examples:

3.2. Example. The function $f(x) = (x^3 - 1)/(x - 1)$ is at first not defined at $x = 1$. But for x close to 1, nothing really bad happens. We can evaluate the function at points closer and closer to 1 and get closer and closer to 3. We say $\lim_{x\to 1} f(x) = 3$. Indeed, you might have noticed already that $f(x) = x^2 + x + 1$ by factoring out $(x - 1)$. While initially not defined at $x = 1$, there is a natural value $b = 3$ we can assign for $f(3)$ so that the graph continues nicely through that point.

3.3. Definition. We write $x \to a$ to indicate that x **approaches** a. This approach can be from either side, **from the left** $x \to a^-$ or **from the right** $x \to a^+$. A function $f(x)$ has a **limit** at a point a if there exists a unique b such that $f(x) \to b$ for $x \to a$. We write $\lim_{x\to a} f(x) = b$ if the limit exists and if it is the same value b, when approaching from either side. [1]

3.4. Example. The **sinc function** $f(x) = \sin(x)/x$ is called $\text{sinc}(x)$. It is not defined at $x = 0$ at first. It appears naturally in geometry as a quotient between the length of a side of a right angle triangle and an arc length of a sector which contains it. We will look at this function a lot also later on and show that the limit of $f(x)$ exists for $x \to 0$. This fact is important.

Fundamental theorem of trigonometry. $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

[1] Technical: for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - b| < \epsilon$.

3.5. Example. The function $f(x) = x/|x|$ is 1 if $x > 0$ and -1 if $x < 0$. It is not defined at $x = 0$ and there is no way to assign a value b at $x = 0$ in such a way that $\lim_{x\to 0} f(x) = b$. One could define $f(0) = 0$ and call the function the *sign* **function**. It is defined everywhere but it is not continuous at 0 as it jumps. We look at continuity in the next lecture.

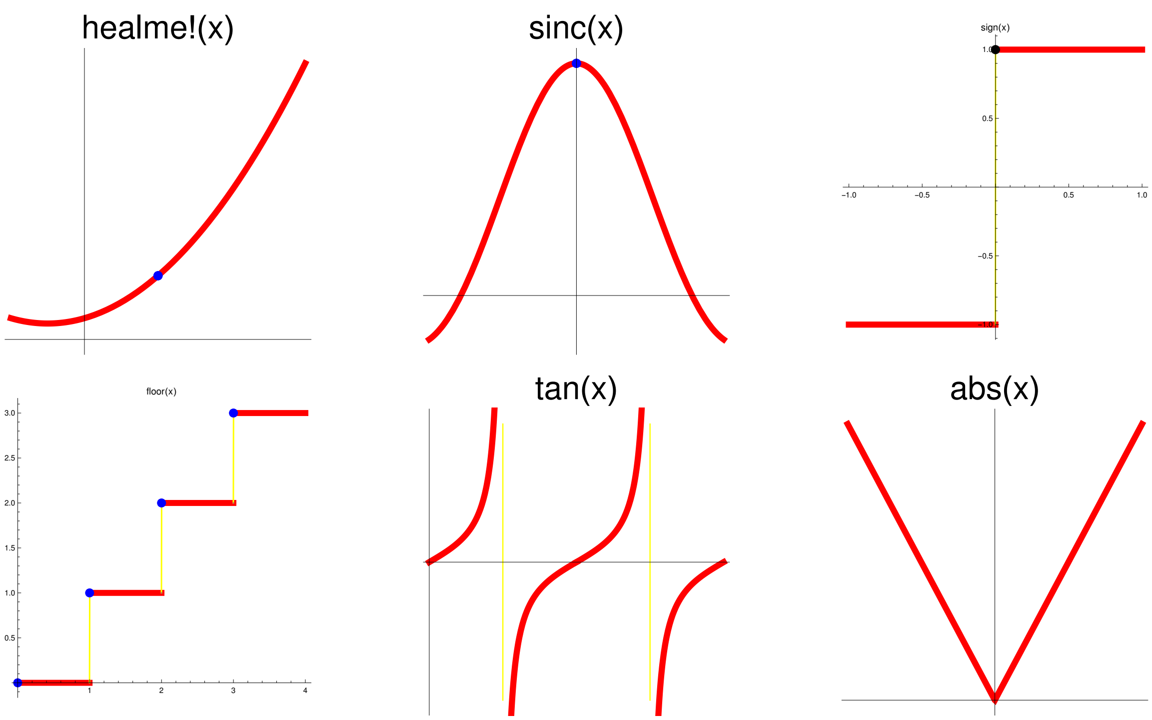


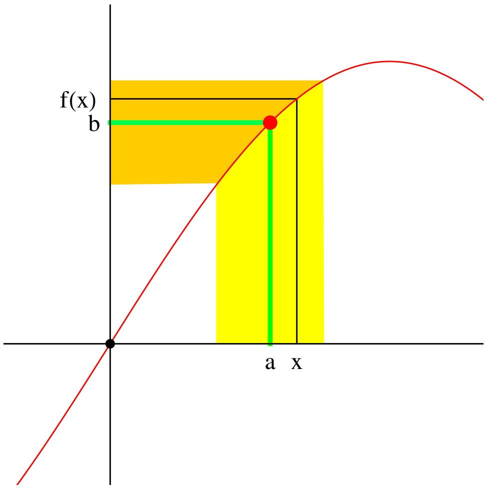
Figure: We see the graphs of $f(x) = (x^3 - 1)/(x - 1)$, the sinc function $\text{sinc}(x) = \sin(x)/x$, the sign function $\text{sign}(x) = x/|x|$, the floor function $\text{floor}(x)$ giving the largest integer smaller or equal to x, the tan function and the **absolute value function** $\text{abs}(x) = |x|$.

3.6. Example. The function $f(x) = \cos(x^2)/(x^4 + 1)$ has the property that $f(x)$ approaches 1 if x approaches 0. To evaluate functions at 0, there was no need to take a limit because $x^4 + 1$ is never zero. The function is everywhere defined. Actually, most functions are nice in the sense that we do not have have to worry about limits at most points. In the overwhelming cases of real applications we only have to worry about limits when the function involves division by 0. For example $f(x) = (x^4+x^2+1)/x$ needs to be investigated more carefully at $x = 0$. You see for example that for $x = 1/1000$, the function is slightly larger than 1000. We can simplify it to $x^3 + x + 1/x$ for $x \neq 0$. There is no limit $\lim_{x\to 0} f(x)$ because $1/x$ has no limit.

3.7. Example. Also, for sin and cos, the limit $\lim_{x\to a} f(x) = f(a)$ is defined. This extends to **trigonometric polynomials** like $\sin(3x) + \cos(5x)$. The function $\tan(x)$ however has no limit at $x = \pi/2$. No finite value b can be found so that $\tan(\pi/2+h) \to b$ for $h \to 0$. This is due to the fact that $\cos(x)$ is zero at $\pi/2$.

3.8. Example. The **cube root** function $f(x) = x^{1/3}$ is defined for all x and even $x = 0$. For the square root function $f(x) = \sqrt{x}$ we have to be aware that for $x < -0$, it is not defined. The domain of the is function is the positive real axis.

Why do we worry about limits? One of the main reasons will is that we will soon define the derivative and integral using limits. A second reason is that limits of polynomials lead to function like the exponential function or logarithm function. An other reason is that one can use limits to define numbers like $\pi = 3.1415926\dots$. In the next lecture, we also look at the important concept of continuity which refers to limits.



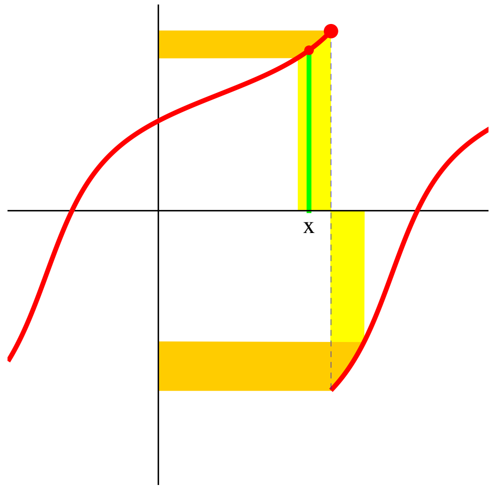


Figure: To the left we see a case, where the limit exists at $x = a$. If x approaches a then $f(x)$ approaches b. To the right we see the function $f(x) = \arctan(\tan(x) + 1)$, where arctan is the inverse of tan. The limit does not exist for $a = \pi/2$. If we approach a from the right, we get the limit $-\pi/2$. From the left, we get the limit $f(\pi/2) = \pi/2$. Note that f is not defined at $x = \pi/2$ because $\tan(x)$ becomes infinite there.

Example: Determine from the following functions whether the limits $\lim_{x\to 0} f(x)$ exist. If it does, find it.

a) $f(x) = \cos(x)/\cos(2x)$	b) $f(x) = \tan(x)/x$
c) $f(x) = (x^2 - x)/(x - 1)$	d) $f(x) = (x^4 - 1)/(x^2 - 1)$
e) $f(x) = (x + 1)/(x - 1)$	f) $f(x) = x/\sin(x)$
g) $f(x) = 5x/\sin(6x)$	h) $f(x) = \sin(x)/x^2$
i) $f(x) = \sin(x)/\sin(2x)$	j) $f(x) = \exp(x)/x$

3.9. The following properties hold for limits:

$\lim_{x\to a} f(x) = b$ and $\lim_{x\to a} g(x) = c$ implies $\lim_{x\to a} f(x) + g(x) = b + c$.
$\lim_{x\to a} f(x) = b$ and $\lim_{x\to a} g(x) = c$ implies $\lim_{x\to a} f(x) \cdot g(x) = b \cdot c$.
$\lim_{x\to a} f(x) = b$ and $\lim_{x\to a} g(x) = c \neq 0$ implies $\lim_{x\to a} f(x)/g(x) = b/c$.

3.10. This implies we can sum up and multiply or divide functions which have limits: Examples: Polynomials like $x^5 - 2x + 6$ or trig polynomials like $\sin(3x) + \cos(5x)$ have limits everywhere. Rational functions like $(x^2 - 1)/(x^2 + 1)$ have limits everywhere if the denominator has no roots. Functions like $\cos^2(x)\tan(x)/\sin(x)$ can be healed by simplification. Prototype functions like $\sin(x)/x$ have limits everywhere.

Homework

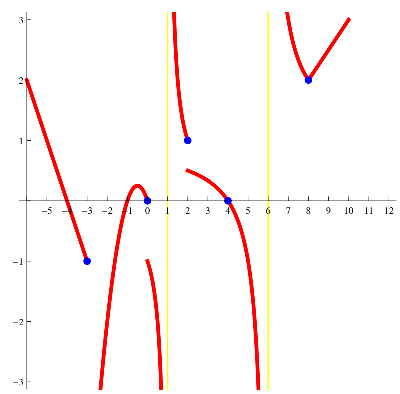
Problem 3.1: Find the limits $x \to 0$. You can use what we have established about $\mathrm{sinc}(x)$.
a) $f(x) = \sin(13x)/x$ b) $f(x) = (x^6 - 1)/(x - 1)$
c) $f(x) = \sin^2(9x)/x^2$, d) $f(x) = \sin(11x)/\sin(7x)$

Problem 3.2: a) Graph of the function

$$f(x) = \frac{(1 - \cos(x))}{x^2} .$$

b) Where is the function f defined? Can you find the limit at the places, where it is not defined? Hint: **remember double angle formulas**
c) Verify that $f(x) = \exp_h(x) = (1 + h)^{x/h}$ satisfies $[f(x + h) - f(x)]/h = f(x)$.
Remark. We define $e^x = \exp(x) = \lim_{h \to 0} \exp_h(x)$.

Problem 3.3: Find all points x at which the function given in the picture has no limits.



Problem 3.4: Find the limits for $x \to 1$:
a) $f(x) = (x^2 - 2x + 1)/(x - 1)$, b) $f(x) = \frac{\sin((x-1))}{(x-1)} \frac{2^x}{5^x}$.
c) $f(x) = \sin^2(x - 1)/(x^2 - 2x + 1)$, d) $f(x) = \frac{\sin(\sin(x))}{\sin(x)}$.

Problem 3.5: We explore in this problem the limit of the function $f(x) = x^x$ if $x \to 0$. Write a short paragraph about it. It should involve some experiments and cases. Can we find a limit in general? Take a calculator or use Wolfram α and experiment. What do you see when $x \to 0$? Can you find an explanation for your experiments?

INTRODUCTION TO CALCULUS

MATH 1A

Unit 4: Continuity

Lecture

4.1.

> **Definition:** A **function** f is **continuous** at a point x_0 if a value $f(x_0)$ can be found such that $f(x) \to f(x_0)$ for $x \to x_0$. A function f is **continuous on** $[a, b]$ if it is continuous for every point x in the interval $[a, b]$.

4.2. In the interior (a, b), the limit needs to exist both from the right and from the left. Intuitively, a function is continuous if one can **draw the graph of the function without lifting the pencil**. Continuity means that small changes in x results in small changes of $f(x)$. Some functions like $(x^2 - 1)/(x - 1)$ or $\sin(x)/x$ need to have function values filled in to become continuous.

4.3. Example. Any polynomial like x^3 or trig functions like $\cos(x), \sin(x), \exp(x)$ for example are continuous. Also the **sum and products** of continuous functions is continuous. For example, $x^5 + \sin(x^3 + e^x)$ is continuous everywhere. We can also **compose** continuous functions like $\exp(\sin(x))$ and still get a continuous function.

4.4. The function $f(x) = 1/x$ is continuous except at $x = 0$. It is a prototype with a **pole discontinuity** at $x = 0$. One can draw a **vertical asymptote**. The **division by zero** kills continuity. Remember however that this can be salvaged in some cases like $f(x) = \sin(x)/x$ which is continuous everywhere. The function can be healed at 0 even so it was at first not defined at $x = 0$.

4.5. The logarithm function $f(x) = \log |x|$ is continuous for all $x \neq 0$. It is not continuous at 0 because $f(x) \to -\infty$ for $|x| \to 0$. It might surprise you that $f(x) = (1 - x^2)/\log |x|$ can be extended to a continuous function. It is not defined at first at $x = 0$ as $\log |0| = -\infty$. It is also not defined at $x = 1$ or $x = -1$ at first because $\log(1) = 0$. But in both cases, we can heal it and see $f(1) = f(-1) = 0$. The value $f(0) = 0$ is easier to see, but filling in the value $f(1) = f(-1) = -2$ is less obvious. We will learn later to heal the function at these two points. It will need hospitalization.

4.6. The **co-secant function** $\csc(x) = 1/\sin(x)$ is not continuous at $x = 0, x = \pi, x = 2\pi$ and more generally for any multiple of π. It has poles there because $\sin(x)$ is zero at those points and because we divide by zero at such points. The function $\cot(x) = \cos(x)/\sin(x)$ shares the same discontinuity points as $\csc(x)$.

4.7. The function $f(x) = \sin(\pi/x)$ is continuous everywhere except at $x = 0$. It is a prototype of a function which is not continuous due to **oscillation**. We can approach $x = 0$ in ways that $f(x_n) = 1$ and such that $f(z_n) = -1$. Just pick $x_n = 2/(4k+1)$ or $z_n = 2/(4k-1)$.

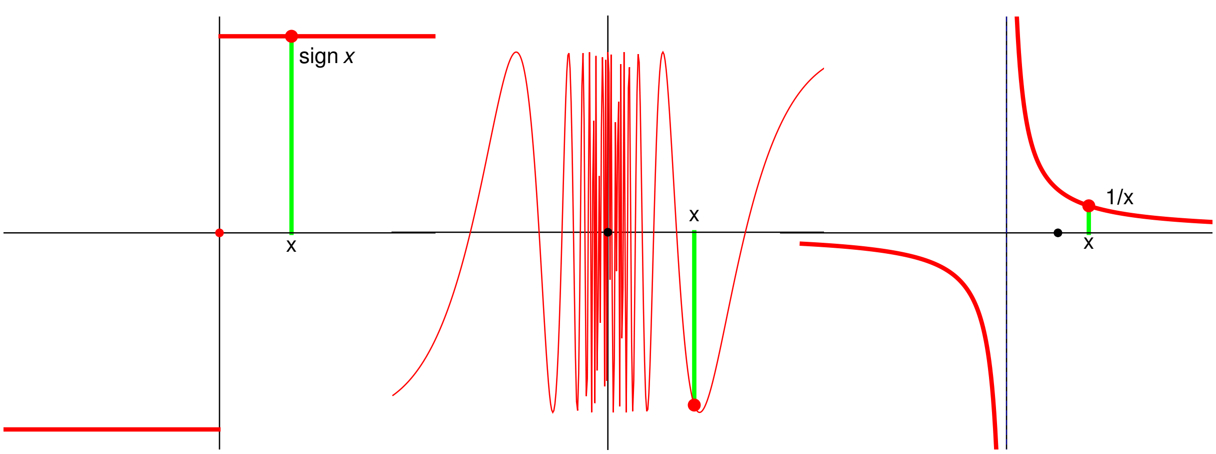
4.8. The **signum function** $f(x) = \text{sign}(x) = \begin{bmatrix} 1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{bmatrix}$ is not continuous at 0.
It is a prototype of a function with a **jump** discontinuity at 0. There is no way we can make this continuous at 0

Rules:
a) If f and g are continuous, then $f + g$ is continuous.
b) If f and g are continuous, then $f * g$ is continuous.
c) If f and g are continuous and if $g \neq 0$ everywhere, then f/g is continuous.
d) If f and g are continuous, then $f \circ g(x) = f(g(x))$ is continuous.

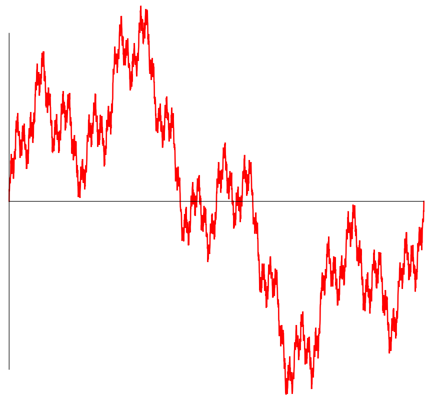
Examples. a) $f(x) = \sqrt{x^2+1}$ is continuous everywhere on the real line. b) $f(x) = \cos(x) + \sin(x)$ is continuous everywhere. c) $f(x) = \log(|x|)$ is continuous everywhere except at 0. d) $f(x) = \sin(\pi x)/\log|x^4|$ is continuous at $x = 0$. Is it continuous everywhere?

Example: The function $f(x) = [\sin(x+h) - \sin(x)]/h$ is continuous for every parameter $h > 0$. We will see soon what happens when h becomes smaller and smaller and that the continuity will never deteriorate but Indeed, we will see $f(x)$ will for smaller and smaller h get to the cos function.

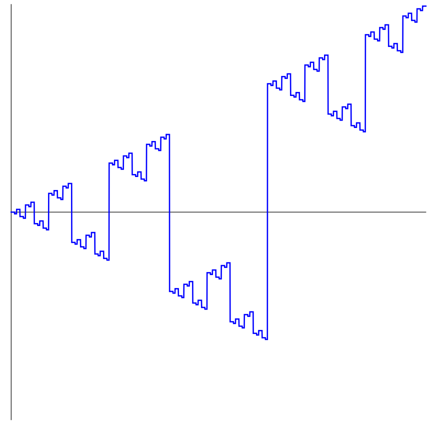
4.9. There are three major reasons, why a function can be not continuous at a point: it can **jump**, **oscillate** or **escape** to infinity. Here are the prototype examples. We will look at more during the lecture.



4.10. Why do we like continuity? One reason important in applications is that "continuity provides stability and some sort of predictability." Discontinuities are usually associated to catastrophes. Discontinuities happen typically, if something breaks.



This Weierstrass function is believed to be a fractal an object of dimension between 1 and 2. But it is continuous.



This function is discontinuous at every point. The vertical connection lines put for clarity are not part of the graph.

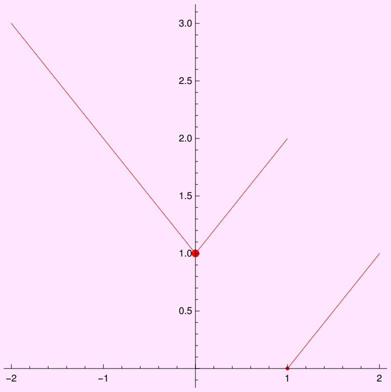
4.11. Continuity will be useful when finding maxima and minima. A continuous function on an interval $[a, b]$ has a maximum and minimum. We will see in the next hour that if a continuous function is negative at some place and positive at an other, there is a point between, where it is zero. Being able to find solutions to equations $f(x) = 0$ is important and much more difficult, if f not continuous.

4.12. Problem: Determine for each of the following functions, where discontinuities appear:
a) $f(x) = \log(|x^2 - 1|)$
b) $f(x) = \sin(\cos(\pi/x))$
c) $f(x) = \cot(x) + \tan(x) + x^4$
d) $f(x) = (x^2 + 2x + 1)/(x + 10 + (x - 1)^2/(x - 1)$
e) $f(x) = \frac{x^2-4x}{x}$

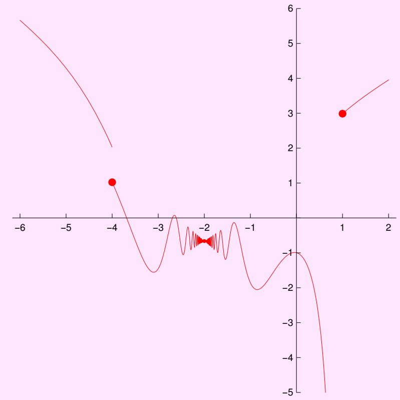
Homework

Problem 4.1: Which functions are continuous everywhere? Remember that we declare functions also continuous if one can fix "broken places" by assigning a value to an initially not defined place. Examples are $\sin(x)/x$ or $(x^2 - 1)/(x - 1)$ which are continuous everywhere when fixed.
a) $\text{sinc}(5x) + \sin(x)/(2 + \sin^2(x)) + (x^3 - 1)/(x - 1)$, b) $\sin(\tan(x))$
c) $\tan(\sin(x)) + \frac{x^2+5x+x^4}{x-3}$
d) $\text{sign}(x)/x$

Problem 4.2: On which intervals are the following functions continuous? You do not have to worry about end points.



a)



b)

Problem 4.3: Either do the following three problems a),b),c):
a) Construct a function which has a jump discontinuity and an escape to infinity.
b) Find a function which has an oscillatory discontinuity and an escape to infinity.
c) Find a function which has a jump discontinuity as well as an oscillatory discontinuity.
or shoot down the problem with one strike:
Find a function which has a jump discontinuity, a pole and an oscillatory discontinuity all at the same time.

Problem 4.4: Heal the following functions to make them continuous everywhere or state that all hope is lost and that no healing is possible:
a) $(x^3 - 27)/(x^2 - 6x + 9)$
b) $\sin(x)(x^7 + x^5)/(x^2 + 3)$
c) $((\sin(x))^3 - \sin(x))/(\cos(x)\sin(x))$.
d) $(x^4 + 4x^3 + 6x^2 + 4x + 1)/(x^3 + 3x^2 + 3x + 1)$
e) $(x^{70} - 1)/(x^{10} - 1)$

Problem 4.5: Are the following function continuous? Break the functions up into simpler functions and analyze each. If you are not sure, experiment by plotting the functions.
a) $\sin(\frac{1}{4+\sin(x)\cos(x)}) + |\cos(x)| + \frac{\sin(x)}{x} + x^5 + x^3 + 1 - \frac{13}{\exp(x)}$.
b) $\frac{7}{\log|x|} + 5x^{77} - \cos(\sin(\cos(x))) - \exp(\log(\exp(x)))$

INTRODUCTION TO CALCULUS

MATH 1A

Unit 5: Intermediate value theorem

Lecture

5.1. Finding solutions to $g(x) = h(x)$ is equivalent to find solutions $f(x) = g(x) - h(x) = 0$.

Definition: If $f(a) = 0$, then a is called a **root** of f. For $f(x) = \sin(x)$ for example, there are roots at $x = 0, x = \pi$.

5.2. Here are a few examples

Example: Find the roots of $f(x) = 4x + 6$. **Answer**: we set $f(x) = 0$ and solve for x. In this case $4x + 6 = 0$ and so $x = -3/2$.

Example: Find the roots of $f(x) = x^2 + 2x + 1$. **Answer**: Because $f(x) = (x + 1)^2$ the function has a root at $x = -1$. In general $x^2 + bx + c$ has roots.

Example: Find the roots of $f(x) = (x - 2)(x + 6)(x + 3)$. **Answer:** Since the polynomial is factored already, it is easy to see the roots $x = 2, x = -6, x = -3$.

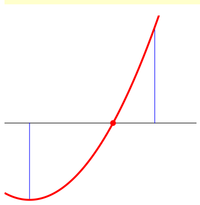
Example: $f(x) = 12 + x - 13x^2 - x^3 + x^4$. Find the roots of f. There is no formula. Just try ($12 = 3 * 4$ is a hint). We see $x = 1, x = -3, x = 4, x = -1$ are the roots.

Example: The function $f(x) = \exp(x)$ does not have any root.

Example: The function $f(x) = \log|x| = \ln|x|$ has roots $x = 1$ and $x = -1$.

Example: $f(x) = 2^x - 16$ has the root $x = 2$.

Intermediate value theorem of Bolzano. If f is continuous on the interval $[a, b]$ and $f(a), f(b)$ have different signs, then there is a root of f in (a, b).

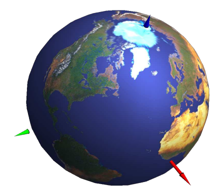


5.3. The proof is constructive: we can assume $f(a) < 0$ and $f(b) > 0$. The other case is similar. Look at $c = (a + b)/2$. If $f(c) < 0$, then take $[c, b]$ as the new interval, otherwise, take $[a, c]$. We get a new root problem on a smaller interval. Repeat the procedure. After n steps, the search is narrowed to an interval $[u_n, v_n]$ of length $2^{-n}(b-a)$. Continuity assures that $f(u_n) - f(v_n) \to 0$ and that $f(u_n), f(v_n)$ have different signs. Both point sequences u_n, v_n converge to a root of f.

Example: Verify that the function $f(x) = x^{17} - x^3 + x^5 + 5x^7 + \sin(x)$ has a root.
Solution. The function goes to $+\infty$ for $x \to \infty$ and to $-\infty$ for $x \to -\infty$. We have for example $f(10000) > 0$ and $f(-1000000) < 0$. Use the theorem.

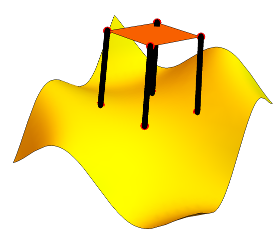
Example: There is a solution to the equation $x^x = 10$. Solution: for $x = 1$ we have $x^x = 1 < 10$ for $x = 10$ we have $x^x = 10^{10} > 10$. Apply the intermediate value theorem to the function $f(x) = x^x - 10$.

Example: **Earth Theorem.** There is a point on the earth, where both temperature and pressure agree with the temperature and pressure on the antipode.



Proof. Draw an arbitrary meridian through the poles and let $f(x)$ be the temperature on that circle, where x is the polar angle. Define the function $g(x) = f(x) - f(x+\pi)$. If g is zero on the north pole, we have found our point. If not, $g(x)$ has different signs on the north and south pole. By the intermediate value theorem, there exists therefore an x, where $g(x) = 0$ and so $f(x) = f(x+\pi)$. For every meridian there is a latitude value $l(y)$ for which the temperature works. Define now $h(y) = l(y) - l(y+\pi)$. This function is continuous. Start with the meridian 0. If $h(0) = 0$ we have found our point. If not, then $h(0)$ and $h(\pi)$ have different signs. By the intermediate value theorem again, h has a root. There, both temperature and pressure agree with the antipode value.

Example: **Wobbly Table Theorem.** On an arbitrary floor, a square table can be turned so that it does not wobble any more.



Proof. The 4 legs ABCD are located on a square in a plane. Let x be the angle of the line AC with with a coordinate axes if we look from above. Given x, the table can be positioned **uniquely**: the center of ABCD is on the z-axes, the legs ABC are on the floor and AC points in the direction x. Let $f(x)$ denote the height of the fourth leg D from the ground. If we find an angle x such that $f(x) = 0$, we have a position where all four legs are on the ground. Assume $f(0)$ is positive. ($f(0) < 0$ is similar.) Tilt the table around the line AC so that the two legs B,D have the same vertical distance h from the ground. Now translate the table down by h. This does not change the angle x nor the center of the table. The two previously hovering legs BD now touch the ground and the two others AC are below. Now rotate around BD so that the third leg C is on the ground. The rotations and lowering procedures have not changed the location of the center of the table nor the direction. This position is the same as if we had turned the table by $\pi/2$. Therefore $f(\pi/2) < 0$. The intermediate value theorem assures that f has a root between 0 and $\pi/2$.

Definition: The last application is a preparation for the derivative which will be introduced next week. Lets call $Df(x) = (f(x+h) - f(x))/h$ the h-**derivative** of f. We will study it more in the next lecture.

Definition: Lets call a point p, where $Df(x) = 0$ a h-**critical point**. Lets call a point a a **local maximum** if $f(a) \geq f(x)$ in an open interval containing a. A **local minimum** is a point a, where $f(a) \leq f(x)$.

Example: As in the first lecture, we can check that the function $f(x) = x(x-h)(x-2h)$ has the derivative $Df(x) = 3x(x-h)$. With the notation $[x]^3 = x(x-h)(x-2h)$ and $[x]^2 = x(x-h)$, we have $D[x]^3 = 3[x]^2$. Since $[x]^2$ has exactly two roots $0, h$, the function $[x]^3$ has 2 critical points.
Example: More generally, for $[x]^{n+1} = x(x-h)(x-2h)\cdots(x-nh)$ we can compute $D[x]^{n+1} = (n+1)D[x]^n$. Because $[x]^n$ has exactly n roots, the function $[x]^{n+1}$ has exactly n critical points. Keep

$$\boxed{D[x]^n = n[x]^{n-1}}$$

in mind! For the usual derivative it will be true $d/dx x^n = nx^{n-1}$.
Example: You can verified that $\exp_h(x) = (1+h)^{x/h}$ satisfies $D\exp_h(x) = \exp_h(x)$. Because this function has no roots and the derivative is the function itself, the function has no h-critical points.

$$\boxed{D\exp(x) = \exp(x)}$$

5.4. The following is an application of the intermediate value theorem and also provides a constructive proof of the **Bolzano extremal value theorem** which we will see later.

Fermat's maximum theorem If f is continuous and has $f(a) = f(b) = f(a+h)$, then f has either a local maximum or local minimum inside the open interval (a,b).

5.5. The argument is to split the interval $[a,b]$ into two $[a,c]$ and $[c,b]$ of the same length. Now, $f(c) - f(a)$ and $f(b) - f(c)$ have different sign so that $g(x) = f(x + h/2) - f(x)$ has different signs $g(a)$ and $g(c)$ at the end points. By the intermediate value theorem there is a root of g in $[a,c]$ and therefore a point x in $[a,c]$ where $f(x) = f(x+h/2)$. This gives a new interval $[a_1, b_1]$ of half the size where the situation $f(a_1) = f(b_1)$ holds. Continuing like this we get a nested sequence of intervals $[a_n, b_n]$ which have size $2^{-n}h$. The limiting point is a maximum or minimum of f.

5.6. "h-critical points" in the limit $h \to 0$ will become "critical point". But h-critical points make sense for all continuous functions. We will need more regularity to take the limit $h \to 0$. The function needs to be **differentiable**. We proved now that any h-critical point leads to either a minimum or maximum.

Homework

Problem 5.1: Find the roots for $-72 - 54x + 35x^2 + 15x^3 - 3x^4 - x^5$. You are told that all roots are integers.

Problem 5.2: Use the intermediate value theorem to verify that $f(x) = x^7 - 6x^6 + 8$ has at least two roots on $[-2, 2]$.

Problem 5.3: The "Queen's gambit" features two fine actors Anya Taylor-Joy and Thomas Brodie-Sangster (both sharing expressive wide eyes). Anya's height is 170 cm, Thomas height is 178 cm. Anya was born April 16, 1996, Thomas was born on May 16, 1990. Anya's and Thomas net worth are both estimated to be 3 Million.
a) Can you argue that there was a moment when Anya's height is exactly half of Thomas height?
b) Can you argue that there was a moment when Anya's age was exactly half the age of Thomas?
c) Can you argue that there as a moment when Anya's fortune was exactly half of Thomas fortune?
Argue with the intermediate value theorem or note a scenario where the statement is false.

Problem 5.4: Argue why there is a solution to
a) $5 - \sin(x) = x$, b) $\exp(7x) = x$, c) $\sin(x) = x^4$.
d) Why does the following argument not work:
The function $f(x) = 1/\cos(x)$ satisfies $f(0) = 1$ and $f(\pi) = -1$. There exists therefore a point x where $f(x) = 0$.
e) Does the function $f(x) = x + \log|\log|x||$ have a root somewhere? Argue with the intermediate value theorem.

Problem 5.5: a) Let $h = 1/2$. Find a h-critical point for the function $f(x) = |x|$. As defined in the text we look for a point for which $[f(x+h) - f(x)]/h = 0$.
b) Verify that for any $h > 0$, the function $f(x) = x^3$ has no h-critical point. There is no x, where $[f(x+h) - f(x)]/h = 0$ is possible.

Unit 6: Fundamental theorem

Lecture

6.1. **Calculus** is a theory of **differentiation** and **integration**. We explore here this concept again in a simple setup and practice differentiation and integration **without taking limits**. We fix a positive constant h and take differences and sums. The fundamental theorem of calculus for $h = 1$ generalizes. We can then differentiate and integrate polynomials, exponentials and trigonometric functions. Later, we will do the same with actual derivatives and integrals. But now, we can work with arbitrary **continuous functions**. The constant h never pops up. You can think of it as something fixed, like the God-given **Planck constant** $1.6 \cdot 10^{-35} m$. In the standard calculus of Newton and Leibniz the limit $h \to 0$ is taken.

Definition: Given $f(x)$, define the **difference quotient**

$$\boxed{Df(x) = \frac{f(x+h)-f(x)}{h}}$$

6.2. If f is continuous then Df is a continuous. For simplicity, we call it "derivative". We keep the positive constant h fixed. As an example, let us take the **constant function** $f(x) = 5$. We get $Df(x) = (f(x+h) - f(x))/h = (5-5)/h = 0$ everywhere. You see that in general, if f is a constant function, then $Df(x) = 0$.

6.3. $f(x) = 3x$. We have $Df(x) = (f(x+h) - f(x))/h = (3(x+h) - 3x)/h$ which is $\boxed{3}$. You see in general that if $f(x) = mx + b$, then $Df(x) = \boxed{m}$.

For $f(x) = c$ we have $Df(x) = 0$. For $f(x) = mx + b$, we have $Df(x) = m$.

6.4. For $f(x) = x^2$ we compute $Df(x) = ((x+h)^2 - x^2)/h = (2hx + h^2)/h = \boxed{2x + h}$.

6.5. For $f(x) = \sqrt{x}$ we compute $Df(x) = (\sqrt{x+h} - \sqrt{x})/h = \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$ which is $1/(\sqrt{x+h} + \sqrt{x})$. For $h \to 0$, we get $1/(2\sqrt{x})$.

6.6. Given a function f, define a new function $Sf(x)$ by summing up all values of $f(jh)$, where $0 \leq jh < x$ with $x = nh$.

Definition: Given $f(x)$ define the **Riemann sum**

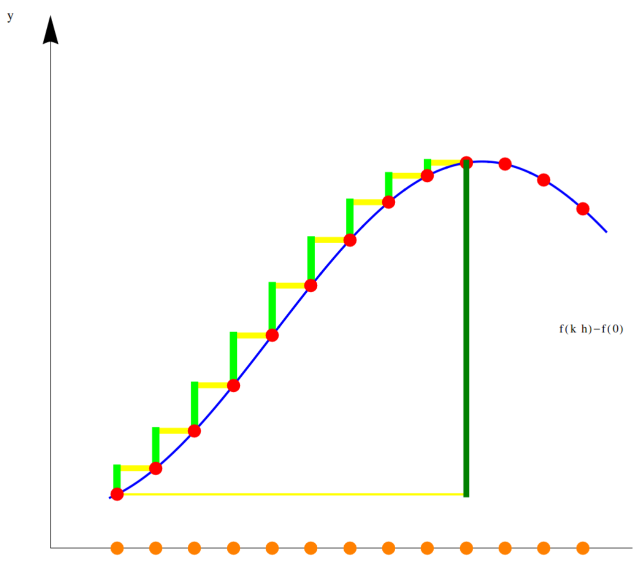
$$\boxed{Sf(x) = h[\ f(0)+f(h)+f(2h)+\cdots+f((n-1)h)\ ]}$$

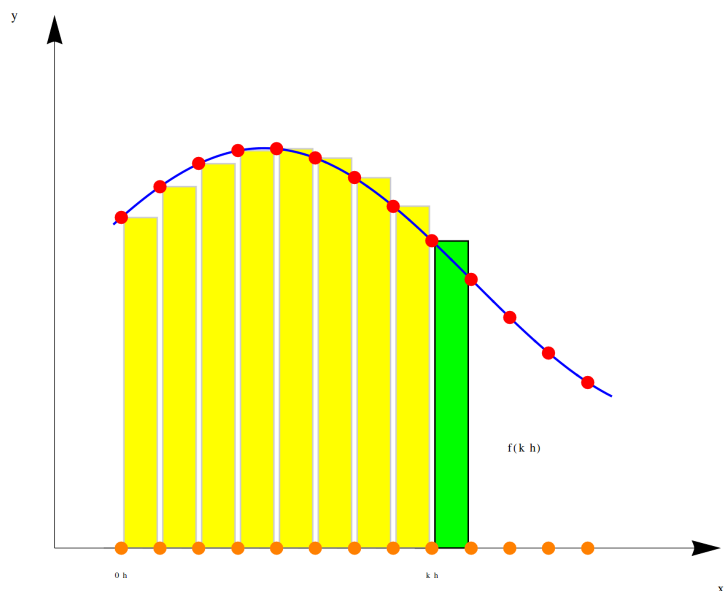
In short hand, we call Sf also the "integral" or "anti-derivative" of f. It will become the integral in the limit $h \to 0$ later in the course.

6.7. Compute $Sf(x)$ for $f(x) = 1$. **Solution**. We have $Sf(x) = 0$ for $x \leq h$, and $Sf(x) = h$ for $h \leq x < 2h$ and $Sf(x) = 2h$ for $2h \leq x < 3h$. In general $S1(jh) = j$ and $S1(x) = kh$ where k is the largest integer such that $kh < x$. The function g grows linearly but grows in quantized steps.

The difference	$Df(x)$	will become the	**derivative**	$f'(x)$.
The sum	$Sf(x)$	will become the	**integral**	$\int_0^x f(t)\ dt$.

Df	means	rise over run	and is close to	the slope	of the	graph of f.
Sf	means	areas of rectangles	and is close to	the area	under the	graph of f.





6.8. Here is the **quantum fundamental theorem of calculus**

Theorem: Sum the differences gives

$$SDf(kh) = f(kh) - f(0)$$

Theorem: Difference the sum gives

$$DSf(kh) = f(kh)$$

Example: For $f(x) = [x]_h^m = x(x-h)(x-2h)...(x-mh+h)$ we have

$$f(x+h)-f(x) = (x(x-h)(x-2h)...(x-kh+2h))\,((x+h)-(x-mh+h)) = [x]^{m-1}hm$$

and so $D[x]_h^m = m[x]_h^{(m-1)}$. We have obtained the important formula $\boxed{D[x]^m = m[x]^{m-1}}$

6.9. This leads to differentiation formulas for **polynomials**. We will leave away the square brackets later to make it look like the calculus we will do later on. In the homework, we already use the usual notation.

6.10. If $f(x) = [x] + [x]^3 + 3[x]^5$ then $Df(x) = 1 + 3[x]^2 + 15[x]^4$. The fundamental theorem allows us to integrate and get $Sf(x) = [x]^2/2 + [x]^4/4 + 3[x]^6/6$.

Definition: Define $\exp_h(x) = (1+h)^{x/h}$. It is equal to 2^x for $h = 1$ and morphs into the function e^x when h goes to zero.

As a rescaled exponential, it is continuous and monotone. Indeed, using rules of the logarithm we can see $\exp_h(x) = e^{x(\log(1+h)/h)} = e^{xA}$. It is actually a classical exponential with some constant A.

6.11. The function $\exp_h(x) = (1 + h)^{x/h}$ has the property that its derivative is the function again (see unit 4). We also have $\exp_h(x + y) = \exp_h(x)\exp_h(y)$. More generally, define $\exp(a \cdot x) = (1+ah)^{x/h}$. It satisfies $\boxed{D\exp_h(a \cdot x) = a\exp_h(a \cdot x)}$ We write a dot because $\exp_h(ax)$ is not equal to $\exp_h(a \cdot x)$. For now, only the differentiation rule for this function is important.

6.12. If a is replaced with ai where $i = \sqrt{-1}$, we have $\exp(1 + ia)(1 + aih)^{x/h}$ and still $D\exp_h^{ai}(x) = ai\exp_h^{ai}(x)$. Taking real and imaginary parts define new **trig functions** $\exp_h^{ai}(x) = \cos_h(a \cdot x) + i\sin_h(a \cdot x)$. These functions are real and morph into the familiar cos and sin functions for $h \to 0$. For any $h > 0$ and any a, we have now $\boxed{D\cos_h(a \cdot x) = -a\sin_h(a \cdot x) \text{ and } D\sin_h(a \cdot x) = a\cos_h(a \cdot x).}$ We will later derive these identities for the usual trig functions.

6.13.

Definition: Define $\log_h(x)$ as the inverse of $\exp_h(x)$ and

$$1/[x + a]_h = D\log_h(x + a).$$

6.14. We have directly from the definition $\boxed{S1/[x+1]_h = \log_h(x+1)}$ As a consequence we can compute things like

$$S\frac{1}{[3x+3]} = \frac{1}{3}S\frac{1}{[x+1]} = \frac{1}{3}\log_h(x+1)\ .$$

More generally $S(1/[x + a]) = \log(x + a) - \log(a)$.

Homework

Use the differentiation and integration rules to find.

Problem 6.1: Find the derivatives $Df(x)$ of the following functions:

a) $f(x) = x^{111} - 3x^{14} + 5x^3 + 1$

b) $f(x) = -x^7 + 8\log(x)$

c) $f(x) = -3x^{13} + 17x^{5/2} - 8x$.

d) $f(x) = \log(x + 1) + 7\sqrt{x}$.

Problem 6.2: Find the integrals $Sf(x)$ of the following functions assuming $Sf(0) = 0$:

a) $f(x) = x^{10} - 8\sqrt{x}$.
b) $f(x) = x^2 - 6x^7 - x$
c) $f(x) = -3x^3 + 17x^2 - 5x$
d) $f(x) = \exp(7x) + \sin(19x)$

Problem 6.3: Find the derivatives $Df(x)$ of the following functions
a) $f(x) = \exp(9 \cdot x + 3) + 2x^6$
b) $f(x) = 8\exp(-3 \cdot x) + 9x^6$
c) $f(x) = \exp(6 \cdot x) + \log(1 + x)$
d) $\log(1 - x^2)$

Problem 6.4: a) Assume $h = 1/100$. Use Wolfram alpha to plot $\cos_h(x)$ and $\sin_h(x)$ on the interval $[-2\pi, 2\pi]$. **Hint.** This means you have to plot the real and imaginary part of $(1 + i/100)^{100x}$. If you enter the expression into Wolfram alpha, it will plot the real and imaginary part.
b) Do the same for $h = 1/1000$. What has changed?

Problem 6.5: a) Write down again on your own that if $f(x) = (1 + ah)^{x/h}$, then verify $Df(x) = af(x)$. (We might have seen this already. Do it again!).
b) Write down again on your own that if $f(x) = x(x-h)(x-2h)(x-3h)$, then $Df(x) = 4x(x-h)(x-2h)$ meaning $D[x]^4 = 4[x]^3$.

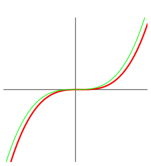
Fundamental theorem of Calculus: $DSf(x) = f(x)$ and $SDf(x) = f(x) - f(0)$.

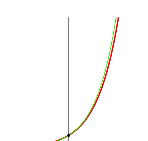
Differentiation rules
$Dx^n = nx^{n-1}$
$De^{a \cdot x} = ae^{a \cdot x}$
$D\cos(a \cdot x) = -a\sin(a \cdot x)$
$D\sin(a \cdot x) = a\cos(a \cdot x)$
$D\log(1 + xa) = a/(1 + ax)$

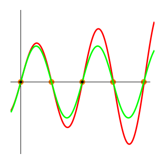
Integration rules (for $x = kh$)
$Sx^n = x^{n+1}/(n+1)$
$Se^{a \cdot x} = (e^{a \cdot x} - 1)/a$
$S\cos(a \cdot x) = \sin(a \cdot x)/a$
$S\sin(a \cdot x) = (1 - \cos(a \cdot x))/a$
$S\frac{1}{1+ax} = \log(1 + ax)/a$

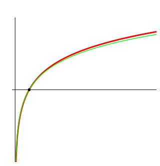
Fermat's extreme value theorem: If $Df(x) = 0$ and f is continuous, then f has a local maximum or minimum in the open interval $(x, x + h)$.

Pictures


$[x]_h^3$ for $h = 0.1$


$\exp_h(x)$ for $h = 0.1$


$\sin_h(x)$ for $h = 0.1$


$\log_h(x)$ for $h = 0.1$

INTRODUCTION TO CALCULUS

MATH 1A

Unit 7: Rate of Change

Lecture

7.1. Given a function f and a constant $h > 0$, we can look at the new function

$$Df(x) = \frac{f(x+h) - f(x)}{h} .$$

It is the **average rate of change** of the function with **step size** h. When changing x to $x+h$ and then $f(x)$ changes to $f(x+h)$. The quotient $Df(x)$ is a **slope** and **"rise over run"**. In this lecture, we take the limit $h \to 0$. It is called the **instantaneous rate of change**. We derive the important formulas $\frac{d}{dx}x^n = nx^{n-1}$, $\frac{d}{dx}\exp(ax) = a\exp(ax)$, $\frac{d}{dx}\sin(ax) = a\cos(ax)$, $\frac{d}{dx}\cos(ax) = -a\sin(ax)$ which we have seen already before in a discrete setting. But now we see them also to in the limit $h \to 0$:

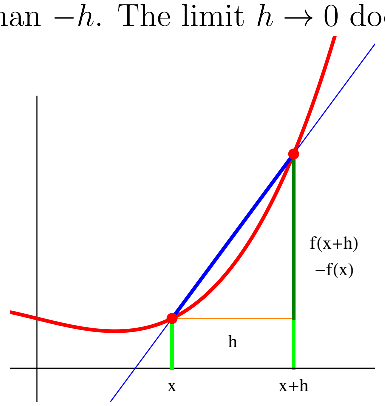
Definition: If the limit $\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ exist, we say f is **differentiable** at the point x. The value is called the **derivative** or **instantaneous rate of change** of the function f at x. We denote the limit also with $f'(x)$.

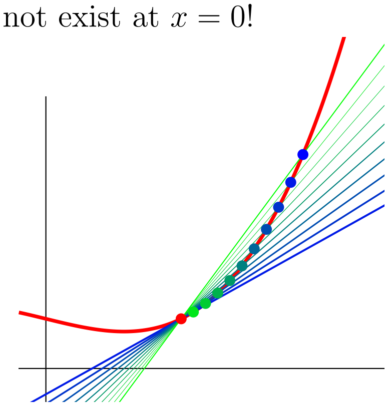
7.2. Example. For $f(x) = 30 - x^2$ we have

$$f(x+h) - f(x) = [30 - (x+h)^2] - [30 - x^2] = -2xh - h^2$$

Dividing this by h gives $-2x - h$. The limit $h \to 0$ gives $-2x$. We have just seen that for $f(x) = x^2$, we get $f'(x) = -2x$. For $x = 3$, this is -6.
Example. For $f(x) = |x|$, we have $(f(x+h) - f(x))/h = 1$ if $x > 0$ and $(f(x+h) - f(x))/h = -1$ if x is smaller than $-h$. The limit $h \to 0$ does not exist at $x = 0$!

The derivative $f'(x)$ has a geometric meaning. It is the slope of the tangent at x. This is an important geometric interpretation. Often, x is "time" and the derivative as the rate of change of the quantity $f(x)$ in time.





For $f(x) = x^n$, we have $f'(x) = nx^{n-1}$.

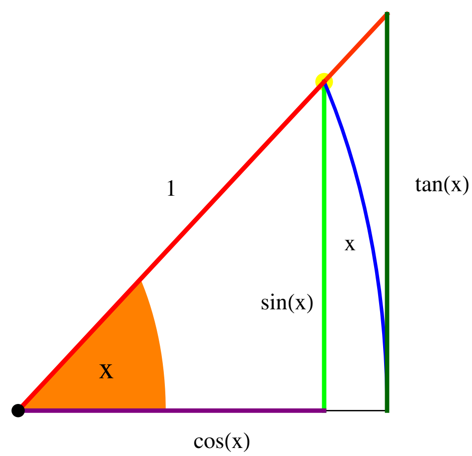
Proof: We can use our discrete calculus set-up and note that $[x]^n$ goes to x^n for $h \to 0$. More traditional is an expansion $f(x+h)-f(x) = (x+h)^n = (x^n+nx^{n-1}h+a_2h^2+\cdots+h^n)-x^n = nx^{n-1}h+a_2h^2+...+h^n)$. If we divide by h, we get $nx^{n-1}+h(a_2+\cdots+h^{n-2})$ for which the limit $h \to 0$ exists: it is nx^{n-1}. This example is important because many functions can be approximated very well with polynomials.

For $f(x) = \sin(x)$ we have $f'(0) = 1$ because the differential quotient is $[f(0+h) - f(0)]/h = \sin(h)/h = \mathrm{sinc}(h)$. We have already mentioned that the limit is 1 before. Lets look at it again geometrically. For all $0 < x < \pi/2$ we have

$$\sin(x) \le x \le \tan(x) .$$

Now divide everything by $\sin(x)$. Because $\tan(x)/\sin(x) = 1/\cos(x) \to 1$ for $x \to 0$, the value of $\mathrm{sinc}(x) = \sin(x)/x$ must go to 1 as $x \to 0$. Renaming the variable x with the variable h, we have verified the **fundamental theorem of trigonometry**

$$\boxed{\lim_{h\to 0} \frac{\sin(h)}{h} = 1}$$



7.3. For $f(x) = \cos(x)$ we have $f'(0) = 0$. To see this, look at $[f(0+h) - f(0)]/h = [\cos(h) - 1]/h$. From $2 - 2\cos(h) = \sin^2(h) + (1 - \cos(h))^2$ which is less than h^2 (geometry!) we have $(1 - \cos(h) \le h^2/2$ so that $(1 - \cos(h))/h \le h/2$. We have now

$$\boxed{\lim_{h\to 0} \frac{(1-\cos(h)}{h} = 0}.$$

The interpretation is that the tangent is **horizontal** for the cos function at $x = 0$.

7.4. From the previous two examples and trig identities we get

$$\cos(x+h)-\cos(x) = \cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x) = \cos(x)(\cos(h)-1)-\sin(x)\sin(h) .$$

Now use the just established $(\cos(h) - 1)/h \to 0$ and the fundamental theorem of trigonometry $\sin(h)/h \to 1$ to see that $[\cos(x + h) - \cos(x)]/h \to -\sin(x)$.

For $f(x) = \cos(ax)$ we have $f'(x) = -a\sin(ax)$.

7.5. Similarly,

$$\sin(x+h)-\sin(x) = \cos(x)\sin(h)+\sin(x)\cos(h)-\sin(x) = \sin(x)(\cos(h)-1)+\cos(x)\sin(h)$$

because $(\cos(h) - 1)/h \to 0$ and $\sin(h)/h \to 1$, we see that $[\sin(x + h) - \sin(x)]/h \to \cos(x)$.

for $f(x) = \sin(ax)$, we have $f'(x) = a\cos(ax)$.

$$e^x = \lim_{n\to\infty} (1 + \frac{1}{n})^{nx}$$

7.6. Like π, the **Euler number** $e = e^1$ is irrational. Here are the first digits: 2.7182818284590452354. If you want to find an approximation, just pick a large n, like $n = 100$ and compute $(1 + 1/n)^n$. For $n = 100$ for example, we see $101^{100}/100^{100}$. We only need to compute the integer 101^{100} and then put a comma after the first digit to get a decent approximation of e.

7.7. To see why the limit exists, verify hat the fractions $A_n = (1 + 1/n)^n$ increase and $B_n = (1 + 1/n)^{(n+1)}$ decrease. Since $B_n/A_n = (1 + 1/n)$ which goes to 1 for $n \to \infty$, the limit exists. The same argument shows that $(1 + 1/n)^{xn} = \exp_{1/n}(x)$ increases and $\exp_{1/n}(x)(1 + 1/n)$ decreases. The limiting function $\exp(x) = e^x$ is called the **exponential function**. Remember that if we write $h = 1/n$, then $(1 + 1/n)^{nx} = \exp_h(x)$ considered earlier in the course. We can sandwich the exponential function between $\exp_h(x)$ and $(1 + h)\exp_h(x)$:

$$\exp_h(x) \leq \exp(x) \leq \exp_h(x)(1 + h), \quad x \geq 0 \ .$$

For $x < 0$, the inequalities are reversed.

7.8. Lets compute the derivative of $f(x) = e^x$ at $x = 0$. **Answer**. We have for $x \leq 1$

$$1 \leq (e^x - 1)/x \leq 1 + x \ .$$

Therefore $f'(0) = 1$. The exponential function has a graph which has slope 1 at $x = 0$. Now, we can get the general case. It follows from $e^{x+h} - e^x = e^x(e^h - 1)$ that the derivative of $\exp(x)$ is $\exp(x)$.

For $f(x) = \exp(ax)$, we have $f'(x) = a\exp(ax)$.

7.9. It follows from the properties of taking limits that $(f(x) + g(x))' = f'(x) + g'(x)$. We also have $(af(x))' = af'(x)$. From this, we can now compute many derivatives

7.10. Find the slope of the tangent of $f(x) = \sin(3x) + 5\cos(10x) + e^{5x}$ at the point $x = 0$. **Solution**: $f'(x) = 3\cos(3x) - 50\sin(10x) + 5e^{5x}$. Now evaluate it at $x = 0$ which is $3 + 0 + 5 = 8$.
Finally, lets mention an example of a function which is not everywhere differentiable.

7.11. The function $f(x) = |x|$ has the properties that $f'(x) = 1$ for $x > 0$ and $f'(x) = -1$ for $x =< 1$. The derivative does not exist at $x = 0$ even so the function is continuous there. You see that the slope of the graph jumps discontinuously at the point $x = 0$.

7.12. For a function which is discontinuous at some point, we don't even attempt to differentiate it there. For example, we would not even try to differentiate $\sin(4/x)$ at $x = 0$ nor $f(x) = 1/x^3$ at $x = 0$ nor $\sin(x)/|x|$ at $x = 0$. Remember these bad guys?

To the end, you might have noticed that in the boxes, more general results have appeared, where x is replaced by ax. We will look at this again but in general, the relation $f'(ax) = af(ax)$ holds ("if you drive twice as fast, you climb twice as fast").

Homework

Problem 7.1: For which of the following functions does the derivative $f'(x)$ exist for every x?

a) $\sin(19x)$ b) $|19\exp(3x)|$ c) $4x + \exp(7x) + 3\sin(45x)$
d) $|\sin(8x)|$ e) $\sin(7/x)$ f) $\exp(-x) + |1 + \cos(15x)|$

Problem 7.2: a) A circle of radius r encloses a disc of area $f(r) = \pi r^2$. Find $\frac{d}{dr} f(r)$. Evaluate the rate of change at $r = 1/10$.
b) The ball of radius r has the volume $f(r) = 4\pi r^3/3$. Find $\frac{d}{dr} f(r))$ at $r = 1/10$ and compare it with the surface area of the sphere bounding the ball.
c) A **hypersphere** of radius r has the **hyper volume** $f(r) = \pi^2 r^4/2$. Find $\frac{d}{dr} f(r)$ and evaluate it at $r = 1/10$.

Problem 7.3: Find the derivatives of the following functions at the point $x = 0$.
a) $f(x) = 7\exp(7x) + 3\sin(22x) + x^7 + 7^x$.
b) $f(x) = 4(x^7 - 1)/(x - 1) + \cos(22x)$. First heal this function.
c) $f(x) = \frac{1+5x+10x^2+10x^3+5x^4+x^5}{x^2+2x+1}$. Also here, first heal!

Problem 7.4: In this problem we compute the derivative of $x^{1/3}$ for $x > 0$. To do so, we have to find the limit

$$\lim_{h \to 0} \frac{(x + h)^{1/3} - x^{1/3}}{h} .$$

Hint: as done in lesson 6 for $\sqrt{x}$ multiply and divide with something and use algebra in particular $(a - b)(a^2 + a * b + b^2) = a^3 - b^3$.

Problem 7.5: Find the derivative of $f(x) = (x+1)^{100} + 100^{x+1}$ at $x = 0$ without actually multiplying out the power. Argue first why $d/dx(x + 1)^n = n(x + 1)^{n-1}$ in general and why $d/dx b^{ax+c} = \log(b) b^{ax+c}$.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 8: Derivative Function

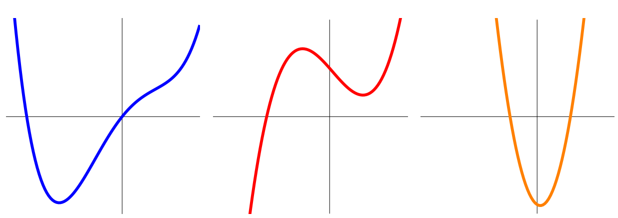
Lecture

8.1. The derivative $f'(x) = \frac{d}{dx}f(x)$ was defined as a limit of $(f(x+h) - f(x))/h$ for $h \to 0$. We have seen that $\frac{d}{dx}x^n = nx^{n-1}$ holds for integer n. We also know already that $\sin'(ax) = a\cos(x), \cos'(ax) = -a\sin(x)$ and $\exp'(ax) = a\exp(ax)$. We can now differentiate already a lot of functions and evaluate the derivative $f'(x)$ at a given point x and compute the slope of the graph of f at x.

8.2. Example: Find the derivative $f'(x)$ of $f(x) = \sin(4x) + \cos(5x) - \sqrt{x} + 1/x + x^4$ and evaluate it at $x = 1$. **Solution**: $f'(x) = 4\cos(4x) - 5\sin(5x) - 1/(2\sqrt{x}) - 1/x^2 + 4x^3$. Plugging in $x = 1$ gives $-\pi - 1/2 - 1 + 4$.

8.3. The differentiation process produces also a rule which assigns to a function f a new function f', the **derivative function**. For example, for $f(x) = \sin(x)$, we get $f'(x) = \cos(x)$. In this lecture, we want to understand the new function and its relation with f. What does it mean if $f'(x) > 0$? What does it mean that $f'(x) < 0$? Do the roots of f tell about f' or do the roots of f' tell about f?

8.4. Here is an example of a function f, its derivative f' and the derivative of the derivative f''. Can you see the relation?



8.5. To understand this, it is good to distinguish intervals, where $f(x)$ is **increasing** or **decreasing**. This are the intervals where $f'(x)$ is positive or negative.

> **Definition:** A function is called **strictly monotonically increasing** on an interval $I = (a, b)$ if $f'(x) > 0$ for all $x \in (a, b)$. It is **strictly monotonically decreasing** if $f'(x) < 0$ for all $x \in (a, b)$.

Monotonically increasing functions "go up" when you "increase x". Lets color that:

Example: Can you find a function f on the interval $[0,1]$ which is bounded $|f(x)| \leq 1$ but such that $f'(x)$ is unbounded?

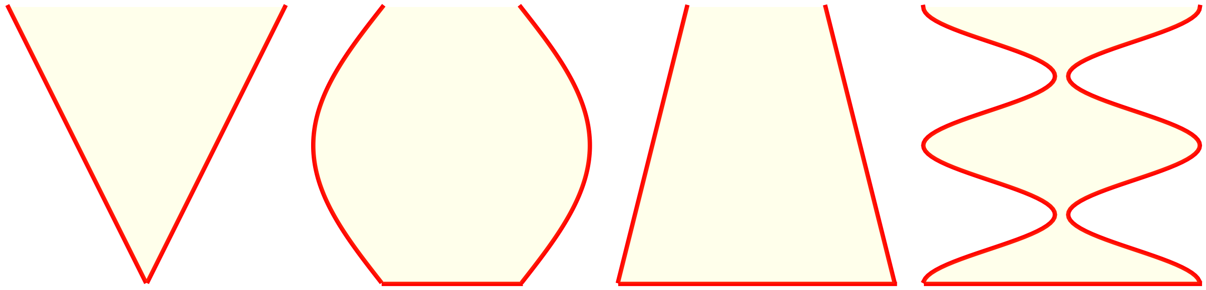
Definition: Given $f(x)$, we can define $g(x) = f'(x)$ and then take the derivative g' of g. This second derivative $f''(x)$ is called the **acceleration**. It measures the rate of change of the tangent slope. For $f(x) = x^4$, for example we have $f''(x) = 12x^2$. If $f''(x) > 0$ on some interval the function is called **concave up**, if $f''(x) < 0$, it is **concave down**.

Example: Find a function f which has the property that its acceleration is constant equal to 6. **Solution.** We have to get a function such that its derivative is $6x$. That works for $3x^2$.

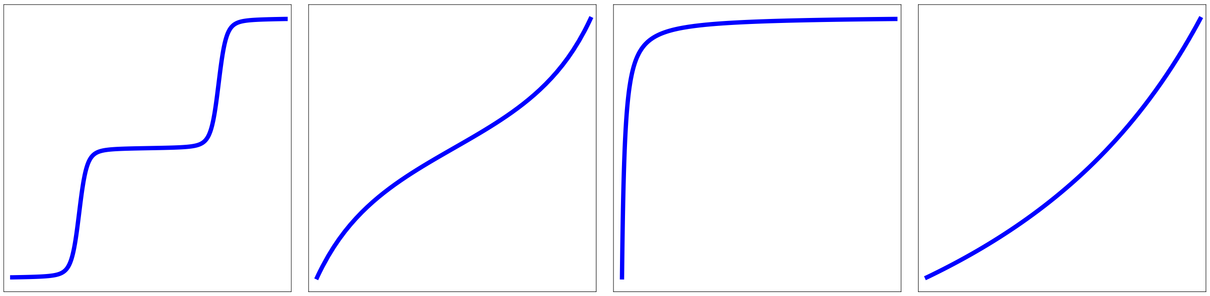
Example: Find a function f which has the property that its acceleration f'' is equal to the negative of f. To do so, try some of the basic functions you know and compute $f'(x), f''(x)$ in each case.

8.6. In the famous **bottle calibration problem**, we fill a circular bottle or glass with constant amount of fluid. Plot the height of the fluid in the bottle at time t. Assume the radius of the bottle is $f(z)$ at height z. Can you find a formula for the height $g(t)$ of the water? This is not so easy. But we can find the rate of change $g'(t)$. Assume for example that f is constant, then the rate of change is constant and the height of the water increases linearly like $g(t) = t$. If the bottle gets wider, then the height of the water increases slower. There is definitely a relation between the rate of change of g and f. Before we look at this more closely, lets try to match the following cases of bottles with the graphs of the functions g qualitatively.

Example: In each of the bottles, we call g the height of the water level at time t, when filling the bottle with a constant stream of water. Can you match each bottle with the right height function?



a) b) d) c)



1) 2) 3) 4)

8.7. The key is to look at $g'(t)$, the rate of change of the height function. Because $[g(t+h)-g(t)]$ times the area πf^2 is a constant times the time difference $h = dt$, we have **bottle calibration formula**

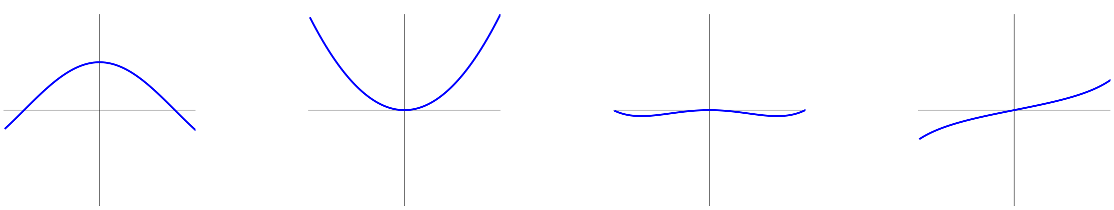
$$g' = \frac{1}{\pi f^2} \ .$$

It relates the derivative function of g with the thickness $f(t)$ of the bottle at height g. No need to learn this. It just explains the story completely. It tells that that if the bottle radius f is large, then the water level increase g' is small and if the bottle radius f is small, then the liquid level change g' is large.

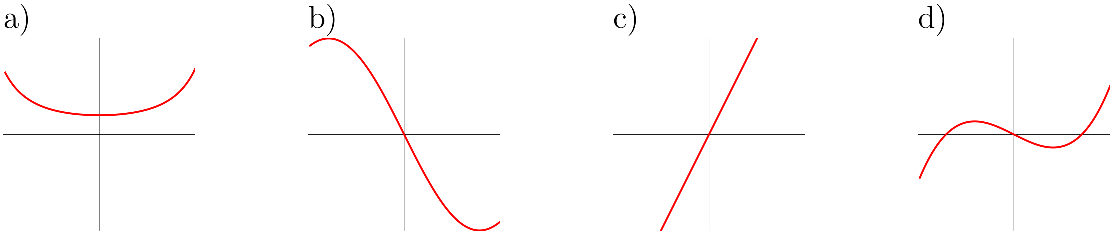
Homework

Problem 8.1: a) Determine on which interval the function $x^4 - x^2$ is monotonically increasing or monotonically decreasing.
b) We write $f'(x)$ for the derivative, $f''(x) = f^{(2)}$ for the second derivative, $f'''(x) = f^{(3)}(x)$ for the third derivative etc. What is the 1000'th derivative of $\sin(x)$?

Problem 8.2: Match the following functions with their derivatives. Explain using monotonicity



a) b) c) d)



1) 2) 3) 4)

Problem 8.3: Draw for the following functions the graph of the function $f(x)$ as well as the graph of its derivative $f'(x)$. You do not have to compute the derivative analytically as a formula here since we do not have all tools yet to compute the derivatives. The derivative function you draw needs to have the right qualitative shape however.
a) The **"To whom the bell tolls"** function

$$f(x) = e^{-x^2}$$

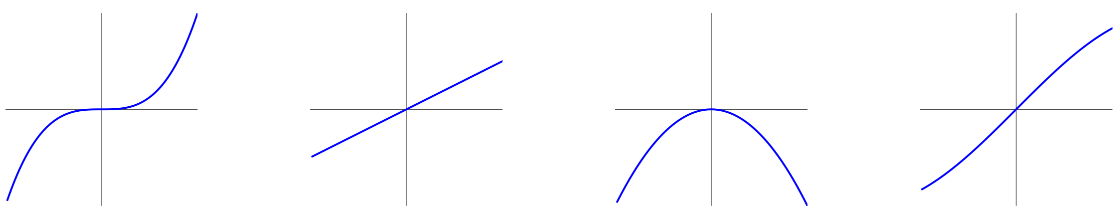
b) The **"Maria Agnesi"** function:

$$f(x) = \frac{1}{1+x^2}$$

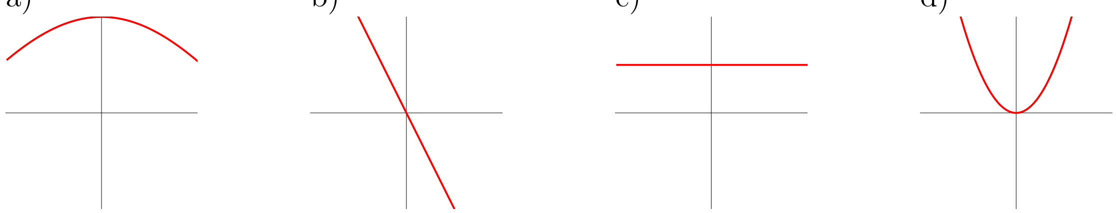
c) The **three gorges function**

$$f(x) = \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1} .$$

Problem 8.4: Match also the following functions with their derivatives. Give short explanations documenting your reasoning in each case.

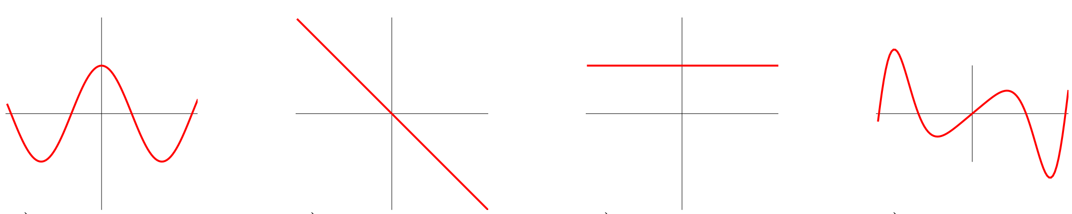


a) b) c) d)



1) 2) 3) 4)

Problem 8.5: Below you the graphs of four different derivative functions $f'(x)$. In each case you are told that $f(0) = 1$. Your task is to draw the function $f(x)$ in each of the cases a),b),c),d). Your picture does not have to be up to scale, but your drawing should display the right features.



d) c) b) a)

INTRODUCTION TO CALCULUS

MATH 1A

Unit 9: Product Rule

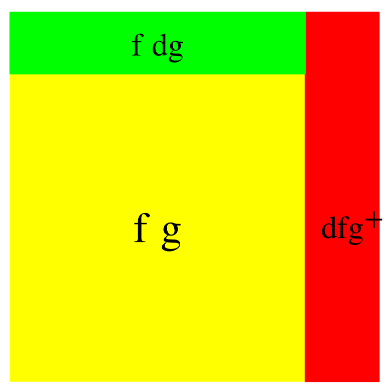
Lecture

9.1. The product rule gives the derivative of a product of functions in terms of the functions and the derivatives of each function. It is also called **Leibniz rule** named after **Gottfried Leibniz**, who found it in 1684. It is important because it allows us to differentiate many more functions. We will be able to compute so the derivative of $f(x) = x\sin(x)$ for example without having to take the limit the limit $\lim(f(x+h) - f(x))/h$. Let us start with an identity which is a discrete Leibniz rule which holds in the **Babylonian calculus** where we do not take limits.

$$f(x+h)g(x+h) - f(x)g(x) = [f(x+h) - f(x)] \cdot g(x+h) + f(x) \cdot [g(x+h) - g(x)] \ .$$

9.2. It can be written as $D(fg) = Dfg^+ + fDg$ with a shifted function $g^+(x) = g(x+h)$ and $Df(x) = [f(x+h) - f(x)]/h$. This **quantum Leibniz rule** can also be seen geometrically: the rectangle of area $(f + df)(g + dg)$ is the union of rectangles with area $f \cdot g$, $f \cdot dg$ and $df \cdot g^+$. Now take the limit $h \to 0$:

$$\frac{[f(x+h)-f(x)]}{h} \cdot g(x+h) \to f'(x) \cdot g(x)$$
$$f(x) \cdot \frac{[g(x+h)-g(x)]}{h} \to f(x) \cdot g'(x)$$



9.3. We get the extraordinarily important **product rule**:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x) \ .$$

Example: Find the derivative function $f'(x)$ for $f(x) = x^3\sin(x)$. **Solution:** We know how to differentiate x^3 and $\sin(x)$ so that $f'(x) = 3x^2\sin(x) + x^3\cos(x)$.

Example: While we know for $f(x) = x^5$ that $f'(x) = 5x^4$, let us compute this with the Leibniz rule. To do so, write $x^5 = x^4 \cdot x$. We have

$$\frac{d}{dx}x^4 = 4x^3, \frac{d}{dx}x = 1 \ .$$

The product rule gives us $f'(x) = 4x^3 \cdot x + x^4 \cdot 1 = 5x^4$. In principle we could be induction prove so the formula $f'(x^n) = nx^{n-1}$.

Example: We look now at a few derivatives related to functions, where we know the answer already but where we can check things using the product formula:

- $\frac{d}{dx}(x^3 \cdot x^5)$
- $\frac{d}{dx}e^{3x}e^{5x}$
- $\frac{d}{dx}\sqrt{x}/\sqrt{x}$
- $\frac{d}{dx}\sin(x)\cos(x)$

9.4. The **quotient rule** allows to differentiate $f(x)/g(x)$. Because we can write this as $f(x) \cdot 1/g(x)$, we need only to differentiate $1/g(x)$. This gives the **reciprocal rule**:

If $g(x) \neq 0$, then

$$\frac{d}{dx}\frac{1}{g(x)} = \frac{-g'(x)}{g(x)^2} \ .$$

9.5. In order to see this, write $h = 1/g$ and differentiate the equation $1 = g(x)h(x)$ on both sides. The product rule gives $0 = g'(x)h(x) + g(x)h'(x)$ so that $h'(x) = -h(x)g'(x)/g(x) = -g'(x)/g^2(x)$.

Example: Find the derivative of $f(x) = 1/x^4$. **Solution**: $f'(x) = -4x^3/x^8 = -4/x^5$. The same computation shows that $\frac{d}{dx}x^n = nx^{n-1}$ holds for all integers n.

The formula $\boxed{\frac{d}{dx}x^n = nx^{n-1}}$ holds for all integers n.

9.6. The **quotient rule** is obtained by applying the product rule to $f(x) \cdot (1/g(x))$ and using the reciprocal rule. This gives the **"Low D High take High D Low - cross the line and square the Low"** rule:

If $g(x) \neq 0$, then

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{[g(x)f'(x) - f(x)g'(x)]}{g^2(x)} \ .$$

Example: Find the derivative of $f(x) = \tan(x)$. **Solution**: because $\tan(x) = \sin(x)/\cos(x)$ we have

$$\tan'(x) = \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} \ .$$

Example: Find the derivative of $f(x) = \frac{2-x}{x^2+x^4+1}$. **Solution**. We apply the quotient rule and get $[(-1)x^2 + x^4 + 1 + (2-x)(2x + 4x^3)]/(x^2 + x^4 + 1)$.

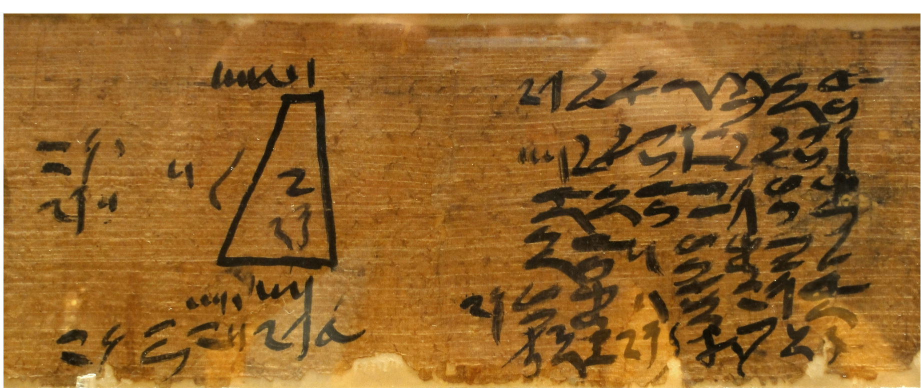
9.7. Here are some more problems with solutions:

Example: Find the second derivative of $\tan(x)$. **Solution**. We have already computed $\tan'(x) = 1/\cos^2(x)$. Differentiate this again with the quotient rule gives $\frac{-\frac{d}{dx}\cos^2(x)}{\cos^4(x)}$. We still have to find the derivative of $\cos^2(x)$. The product rule gives $\cos(x)\sin(x) + \sin(x)\cos(x) = 2\cos(x)\sin(x)$. This gives $2\sin(x)/\cos^3(x)$.

Example: A cylinder has volume $V = \pi r^2 h$, where r is the radius and h is the height. Assume the radius grows like $r(t) = 1+t$ and the height shrinks like $1-\sin(t)$. Does the volume grow or decrease at $t = 0$?

Solution: The volume $V(t) = \pi(1+t)^2(1-\sin(t))$ is a product of two functions $f(t) = \pi(1+t)^2$ and $g(t) = (1-\sin(t)$. We have $f(0) = 1, g'(0) = 2, f'(0) = 2, g(0) = 1$. The product rule gives gives $V'(0) = \pi 1 \cdot (-1) + \pi 2 \cdot 1 = \pi$. The volume increases in volume at first.

9.8. On the **Moscow papyrus** dating back to 1850 BC, the general formula $V = h(a^2 + ab + b^2)/3$ for a truncated pyramid with base length a, roof length b and height h appeared. Assume $h(t) = 1+\sin(t), a(t) = 1+t, b(t) = 1-2t$. Does the volume of the truncated pyramid grow or decrease at first? **Solution**. We could fill in $a(t), b(t), h(t)$ into the formula for V and compute the derivative using the product rule. A bit faster is to write $f(t) = a^2 + ab + b^2 = (1+t)^2 + (1-3t)^2 + (1+t)(1-3t)$ and note $f(0) = 3, f'(0) = -6$ then get from $h(t) = (1+\sin(t))$ the data $h(0) = 1, h'(0) = 1$. So that $V'(0) = (h'(0)f(0) - h(0)f'(0))/3 = (1 \cdot 3 - 1(-6))/3 = -1$. The pyramid shrinks in volume at first.



Example: We pump up a balloon and let it fly. Assume that the thrust increases like t and the resistance decreases like $1/\sqrt{1-t}$ since the balloon gets smaller. The distance traveled is $f(t) = t/\sqrt{1-t}$. Find the velocity $f'(t)$ at time $t = 0$.

Homework

Problem 9.1: Find the derivatives of the following functions, then evaluate at $x = 0$
a) $f(x) = \cos(34x)\tan(47x)$.
b) $f(x) = \cos^3(x)/(1+x)^2$.
c) $f(x) = 3e^x \sin(x)\cos(x)$.
d) $f(x) = 8/(\cos(x) + \sqrt{x+1})$.
e) $f(x) = 6xe^{2x} + 8\tan(3x)$.

Problem 9.2: a) Verify that for $f(x) = g(x)h(x)k(x)l(x)$ the formula $f' = g'hkl + gh'kl + ghk'l + ghkl'$ holds.
b) Verify the following formula for derivative of $f(x) = g(x)^4$ we have $f'(x) = 4g^3(x)g'(x)$. Do not use the chain rule but $f(x) = g(x)g(x)g(x)g(x)$ and a).

Problem 9.3: If $f(x) = \text{sinc}(x) = \sin(x)/x$, find its derivative $g(x) = f'(x)$ and then the derivative of $g(x)$. Then evaluate this at $x = 0$.

Problem 9.4: Find the derivative of

$$\frac{5\sin(x)}{1 + \cos(x) + \frac{x^4}{\sin(x)}}$$

in the limit $x = 0$. Try to do this as effectively as you can and use especially that we evaluate it at a specific point.

Problem 9.5: Here is a preparation for the chain rule, we see in the next unit. But please avoid the chain rule in a) and b) .
a) We have already computed the derivative of $g(x) = \sqrt{x}$ in the last homework. Introduce $f(x) = x^{1/4}$ and apply the product rule to $g(x) = f(x)f(x)$ to get the derivative of f.
b) Use problem 2b) applied to the identity $x = f(x)^4$ to get the derivative of f.
c) If you know the chain rule, use it to get $f'(x)$. If you have not seen the chain rule, no problem, just look it up. We will cover it next time.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 10: Chain rule

Lecture

10.1. In order to take the derivative of a composition of functions like $f(x) = \sin(x^7)$, the product rule does not work. The functions are not multiplied but are "chained" in the sense that we evaluate first x^7 then apply the sin function to it. In order to differentiate, we take the derivative of the x^7 then multiply this with the derivative of the function sin evaluated at x^7. The answer is $7x^6 \cos(x^7)$.

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \ .$$

10.2.

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x) + (g(x+h) - g(x))) - f(g(x))]}{[g(x+h) - g(x)]} \cdot \frac{[g(x+h) - g(x)]}{h} \ .$$

Write $H(x) =$ g(x+h)-g(x) in the first part on the right hand side

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x) + H) - f(g(x))]}{H} \cdot \frac{g(x+h) - g(x)}{h} \ .$$

As $h \to 0$, we also have $H \to 0$ and the first part goes to $f'(g(x))$ and the second factor has $g'(x)$ as a limit.

10.3. Let us look at some examples.

Example: Find the derivative of $f(x) = (4x^2 - 1)^{17}$. **Solution** The inner function is $g(x) = 4x^2 - 1$ with derivative $8x$. We get therefore $f'(x) = 17(4x - 1)^6 \cdot 8x$. **Remark.** We could have expanded out the power $(4x^2 - 1)^{17}$ first and avoided the chain rule. Try it. You will see that the rule of avoiding the **chain rule** is called the **pain rule**.

Example: Find the derivative of $f(x) = \sin(\pi \cos(x))$ at $x = 0$. **Solution**: applying the chain rule gives $\cos(\pi \cos(x)) \cdot (-\pi \sin(x))$.

Example: For linear functions $f(x) = ax + b, g(x) = cx + d$, the chain rule can readily be checked: we have $f(g(x)) = a(cx + d) + b = acx + ad + b$ which has the derivative ac. This agrees with the definition of f times the derivative of g. You can convince you that the chain rule is true also from this example since if you look closely at a point, then the function is close to linear.

10.4. One of the cool applications of the chain rule is that we can compute derivatives of inverse functions:

Example: Find the derivative of the natural logarithm function $\log(x)$. [1] **Solution** Differentiate the identity $\exp(\log(x)) = x$. On the right hand side we have 1. On the left hand side the chain rule gives $\exp(\log(x))\log'(x) = x\log'(x) = 1$. Therefore $\log'(x) = 1/x$.

$\frac{d}{dx}\log(x) = 1/x$.

Definition: Denote by $\arccos(x)$ the inverse of $\cos(x)$ on $[0,\pi]$ and with $\arcsin(x)$ the inverse of $\sin(x)$ on $[-\pi/2, \pi/2]$.

Example: Find the derivative of $\arcsin(x)$. **Solution**. We write $x = \sin(\arcsin(x))$ and differentiate.

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}} .$$

Example: Find the derivative of $\arccos(x)$. **Solution**. We write $x = \cos(\arccos(x))$ and differentiate.

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}} .$$

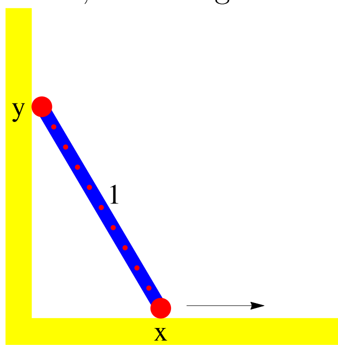
Example: $f(x) = \sin(x^2+3)$. Then $f'(x) = \cos(x^2+3)2x$.

Example: $f(x) = \sin(\sin(\sin(x)))$. Then $f'(x) = \cos(\sin(\sin(x)))\cos(\sin(x))\cos(x)$.
Why is the chain rule called "chain rule". The reason is that we can chain even more functions together.

Example: Let us compute the derivative of $\sin(\sqrt{x^5-1})$ for example. **Solution:** This is a composition of three functions $f(g(h(x)))$, where $h(x) = x^5 - 1$, $g(x) = \sqrt{x}$ and $f(x) = \sin(x)$. The chain rule applied to the function $\sin(x)$ and $\sqrt{x^5-1}$ gives $\cos(\sqrt{x^5-1})\frac{d}{dx}\sqrt{x^5-1}$. Apply now the chain rule again for the derivative on the right hand side.

[1] We always write $\log(x)$ for the natural log. Similarly as $\exp(x) = e^x$, one can also use ln which stands for "logarithmus naturalis". Practically all computer languages like Python, C, Perl, R, Matlab, Mathematica use log. Paul Halmos called "ln" a childish notation which no mathematician ever used.

Example: Here is a famous **falling ladder problem**. A stick of length 1 slides down a wall. How fast does it hit the floor if it slides horizontally on the floor with constant speed? The ladder connects the point $(0, y)$ on the wall with $(x, 0)$ on the floor. We want to express y as a function of x. We have $y = f(x) = \sqrt{1 - x^2}$. Taking the derivative, assuming $x' = 1$ gives $f'(x) = -2x/\sqrt{1 - x^2}$.



In reality, the ladder breaks away from the wall. One can calculate the force of the ladder to the wall. The force becomes zero at the **break-away angle** $\theta = \arcsin((2v^2/(3g))^{2/3})$, where g is the gravitational acceleration and $v = x'$ is the velocity.

Example: For the brave: find the derivative of $f(x) = \cos(\cos(\cos(\cos(\cos(\cos(\cos(x)))))))$.

Example: Take the derivative of $f_3(x) = e^{e^{e^x}}$.

Solution We can also write this as $\exp(\exp(\exp(x))$. The derivative is

$$\exp(\exp(\exp(x))) \exp(\exp(x)) \exp(x) \; .$$

Example: Lets push that to the extreme and differentiate

$$f(x) = \exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(x)))))))))))$$

Here is the poetic formula obtained when running this in Mathematica:

$$\mathrm{F[f\_] := Exp[f]; D[Last[NestList[F, x, 11]], x]}$$

$$\exp\left(e^{e^{e^{e^{e^{e^{e^{e^{e^{e^{x}}}}}}}}}} + e^{e^{e^{e^{e^{e^{e^{e^{e^{x}}}}}}}}} + e^{e^{e^{e^{e^{e^{e^{e^{x}}}}}}}} + e^{e^{e^{e^{e^{e^{e^{x}}}}}}} + e^{e^{e^{e^{e^{e^{x}}}}}} + e^{e^{e^{e^{e^{x}}}}} + e^{e^{e^{e^{x}}}} + e^{e^{e^{x}}} + e^{e^{x}} + e^{x} + x\right)$$

Example: Find the derivative of $1/\sin(x)$ using the quotient rule.

Solution $-\cos(x) \cdot 1/\sin^2(x)$.

Example: Find the derivative of $f(x) = 1/\sin(x)$ using the chain rule.

Solution. The outer function is $f(x) = 1/x$. Therefore $f'(x) = -\cos(x)/\sin^2(x)$.

Homework

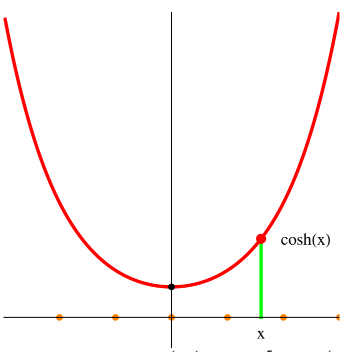
Problem 10.1: Find the derivatives of the following functions:
a) $f(x) = \log(\log(x))$
b) $f(x) = \cot(x^{17})$
c) $f(x) = x/(4 + x^2))$
d) $(\sin(x) + \cos(x))^{-3}$

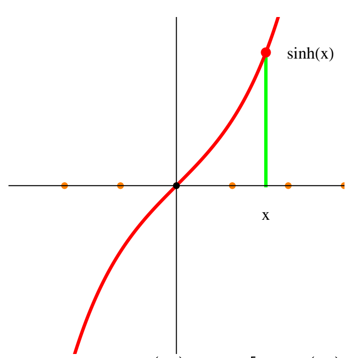
Problem 10.2: Find the derivatives of the following functions at $x = 1$.
a) $\sqrt{x^2 - 1}$
b) $f(x) = -x\log(x)$. (where log is natural log)
c) $(1 + x^3)^{100}$
d) $\sin(\sin(\sin(x)))$

Problem 10.3: a) Find the derivative of $f(x) = 1/x$ by differentiating the identity $xf(x) = 1$ and using the product rule.
b) Find the derivative of $f(x) = \text{arccot}(x)$ by differentiating $\cot(\text{arccot}(x)) = x$ and using the chain rule.

Problem 10.4: a) Find the derivative of $\sqrt{x}$ by differentiating the identity $f(x)^2 = x$ leaving f as it is and solving for $f'(x)$.
b) Find the derivative of $x^{m/n}$ by differentiating the identity $f(x)^n = x^m$ leaving f as it is and solving for $f'(x)$.

Problem 10.5: a) Find the derivative of the inverse $\text{arccosh}(x)$ of $\cosh(x)$ by using the chain rule.
b) Find the derivative of the inverse $\text{arcsinh}(x)$ of $\sinh(x)$ by using the chain rule.





Define $\cosh(x) = [\exp(x) + \exp(-x)]/2$ and $\sinh(x) = [\exp(x) - \exp(-x)]/2$. the **hyperbolic cosine** and **hyperbolic sine**. The cosh function is the shape of a chain hanging at two points. The shape is the hyperbolic cosine. You check $\cosh^2(x) - \sinh^2(x) = 1$.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 11: Critical Points

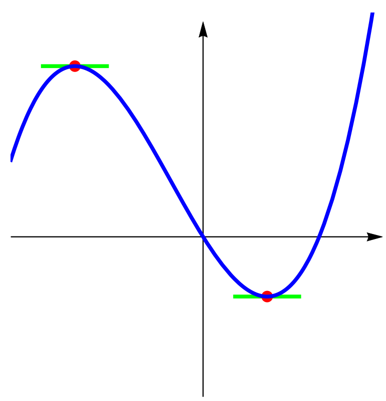
Lecture

11.1. We like to maximize nice quantities and minimize unpleasant ones. Optimizing quantities is also an important principle which nature follows: laws in physics like Newton's law describing the motion of planets, or the Maxwell's equations describing the propagation of light, or the equations written down by Einstein to describe how matter influences geometric space are based on the principle of extremization. An important intuitive insight is that at maxima or minima of a function f, the tangent to the graph must be horizontal. This leads to the following notion for differentiable functions:

Definition: A point x_0 is a **critical point** of f if $f'(x_0) = 0$.

11.2. In some textbooks, critical points also include points, where f' is not defined. Others also include boundary points. [1] We therefore do **not** include boundary points when we make a list of critical points. These points are considered to be outside the domain of definition of f' and we deal with them separately.

Example: Find the critical points of the function $f(x) = x^3 + 3x^2 - 24x$. **Solution:** we compute the derivative as $f'(x) = 3x^2 + 6x - 24$. The roots of f' are $2, -4$.



Definition: A point is called a **local maximum** of f, if there exists an interval $U = (p - a, p + a)$ around p, such that $f(p) \geq f(x)$ for all $x \in U$. A **local minimum** is a local maximum of $-f$. Local maxima and minima together are called **local extrema**.

[1] In all more advanced math textbooks, critical points are defined points where $f' = 0$. Important definitions have to be simple.

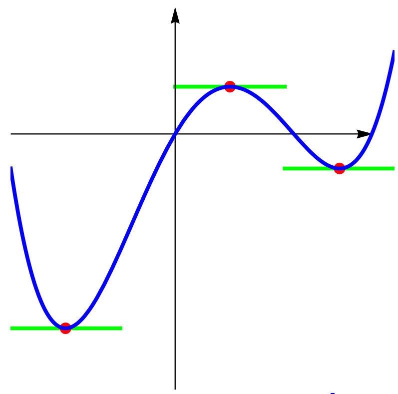
Example: The point $x = 0$ is a local maximum for $f(x) = \cos(x)$. The reason is that $f(0) = 1$ and $f(x) < 1$ nearby.

Example: The point $x = 1$ is a local minimum for $f(x) = (x-1)^2$. The function is zero at $x = 1$ and positive everywhere else.

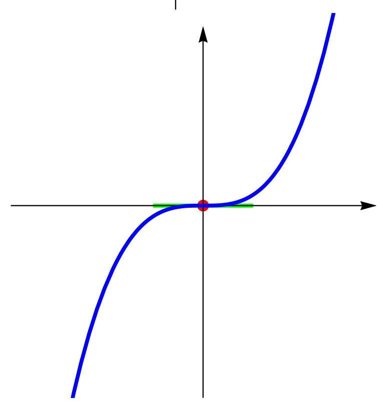
Fermat: If f is differentiable and has a local extremum at x, then $f'(x) = 0$.

11.3. Why is this so? Assume the derivative $f'(x) = c$ is non-zero. We can assume $c > 0$ otherwise replace f with $-f$. By the definition of limits, for some small enough h, we have $f(x+h) - f(x)/h \geq c/2$. But this means $f(x+h) \geq f(x) + hc/2$ and x can not be a local maximum.

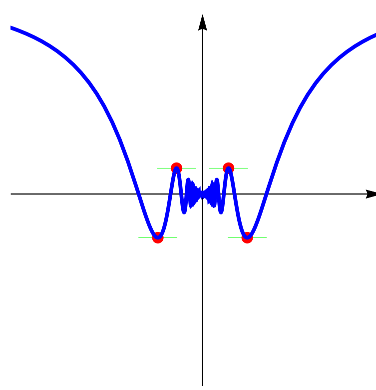
Example: The derivative of $f(x) = 72x - 30x^2 - 8x^3 + 3x^4$ is $f'(x) = 72 - 60x - 24x^2 + 12x^3$. By plugging in integers (calculus teachers like integer roots because students like integer roots!) we can guess the roots $x = 1, x = 3, x = -2$ and see $f'(x) = 12(x-1)(x+2)(x-3)$. The critical points are $1, 3, -2$.



Example: We have already seen that $f'(x) = 0$ does not necessarily imply that x is a local maximum or minimum. The function $f(x) = x^3$ is a counter example. It satisfies $f'(0) = 0$ but 0 is neither a minimum nor maximum there. It is an example of an **inflection point**, which is a point, where the second derivative f'' changes sign.



Example: The function $f(x) = x\sin(1/x)$ is continuous at $x = 0$ but there are infinitely many critical points near 0. The function f is not differentiable at 0, the derivative $\sin(1/x) - \cos(1/x)/x$ not only oscillates like crazy at $x = 0$, it also blows up at $x = 0$.



11.4. If $f''(x) > 0$, then the graph of the function is **concave up**. If $f''(x) < 0$ then the graph of the function is **concave down**.

Second derivative test. If x is a critical point of f and $f''(x) > 0$, then f is a local minimum. If $f''(x) < 0$, then f is a local maximum.

11.5. If $f''(x_0) > 0$ then $f'(x)$ is negative for $x < x_0$ and positive for $f'(x) > x_0$. This means that the function decreases left from the critical point and increases right from the critical point. The point x_0 is a local minimum. Similarly, if $f''(x_0) < 0$ then $f'(x)$ is positive for $x < x_0$ and $f'(x)$ is positive for $x > x_0$. This means that the function increases left from the critical point and increases right from the critical point. The point is a local maximum.

Example: The function $f(x) = x^2$ has one critical point at $x = 0$. Its second derivative is 2 there.

Example: Find the local maxima and minima of the function $f(x) = x^3 - 3x$ using the second derivative test. **Solution:** $f'(x) = 3x^2 - 3$ has the roots $1, -1$. The second derivative $f''(x) = 6x$ is negative at $x = -1$ and positive at $x = 1$. The point $x = -1$ is therefore a local maximum and the point $x = 1$ is a local minimum.

Example: Find the local maxima and minima of the function $f(x) = \cos(\pi x)$ using the second derivative test.

Example: For the function $f(x) = x^6 - x^4$, the second derivative test is inconclusive at $x = 0$. Can you nevertheless see what the nature of the critical point 0 is?

Example: Also for the function $f(x) = x^4$, the second derivative test is inconclusive at $x = 0$. The second derivative is zero. Can you nevertheless see whether the critical point 0 is a local maximum or a local minimum?

Let us look at an example, where we can review the chain rule.

Example: Find the critical points of $f(x) = 4\arctan(x) + x^2$. **Solution**. The derivative is

$$f'(x) = \frac{4}{1+x^2} + 2x = \frac{2x + 2x^3 + 4}{1+x^2} .$$

We see that $x = -1$ is a critical point. There are no other roots of $2x + 2x^3 + 4 = 0$. How did we get the derivative of arctan again? Differentiate:$\tan(\arctan(x)) = x$ and write $u = \arctan(x)$:

$$\frac{1}{\cos^2(\text{ u })} \arctan'(x) = 1 .$$

Use the identity $1 + \tan^2(\text{ u }) = 1/\cos^2(\text{ u })$ to write this as

$$(1 + \tan^2(\text{ u }))\arctan'(x) = 1 .$$

But $\tan(u) = \tan(\arctan(x)) = x$ so that $\tan^2(u) = x^2$. And we have $(1+x^2)\arctan'(x) = 1$. Solving for $\arctan'(x)$ gives $\arctan'(x) = \frac{1}{1+x^2}$.

Example: Finally, let us look at the function $\cosh(x) = (e^x + e^{-x})/2$. We have seen this function already in the last homework. It is the chain curve, because its graph is the shape of a chain. Its sibling is $\sinh(x)$, spelled “sinch” like “grinch”.

Homework

Problem 11.1: Find all critical points for the following functions. If there are infinitely many, indicate their structure. For $f(x) = \cos(x)$ for example, the critical points can be written as $\pi/2 + k\pi$, where k is an integer.
a) $f(x) = x^8 - 4x^2$.
b) $f(x) = 7 + 6\sin(2\pi x) + 3$
c) $f(x) = 3\exp(-x^2)x^2$.
d) $f(x) = 2\sin(\cos(\pi x))$

Problem 11.2: Find all the critical points and use the second derivative test to determine whether they are maxima or minima.
a) $f(x) = 2/(1 + x^2)$
b) $f(x) = x^2 - 2x + 1$.
c) $f(x) = 4x\tan(x)$, where $-\pi/2 < x < \pi/2$ d) $f(x) = -x\log(x)$, where $x > 0$.

Problem 11.3: a) Verify that any cubic equation $f(x) = x^3+ax^2+bx+c$ always has an inflection point, a point where $f''(x)$ changes sign.
b) Where is the inflection point for $a = 2, b = 1, c = 3$? Is this point also a critical point? Is it a maximum or minimum?

Problem 11.4: Depending on c, the function $f(x) = x^4 - cx^2$ has either one or three critical points. Use the second derivative test to decide: a) For $c = 1$, find and determine the nature of the critical points.
b) For $c = -1$, find and determine the nature of the critical points.

Problem 11.5: a) Find your own function which has exactly one local maximum and local minimum on the real line.
b) Engineer a concrete function which has exactly 2 local maxima and 1 local minimum.
c) Find a not necessarily continuous function on the real line with 2 local maxima and no local minimum. If it works, give one. If it does not work, give a reason why it does not work.
d) Is there a differentiable function for which we have 2 local maxima and no local minimum? If it works, give one. If it does not work, give a reason why it does not work.

Unit 12: Global extrema

Lecture

12.1. In this lecture we are interested in global maxima. These are points where the function is maximal overall. These **global extrema** can occur either at critical points of f or at the boundary of the domain, where both f and f' are defined.

> **Definition:** A point a is called a **global maximum** of f if $f(a) \geq f(x)$ for all x. It is called a **global minimum** of f if $f(a) \leq f(x)$ for all x.

12.2. How do we find global maxima? The answer is simple: make a list of all local extrema and boundary points, then pick the largest. Global maxima or minima do not need to exist however. The function $f(x) = x^2$ has a global minimum at $x = 0$ but no global maximum. The function $f(x) = x^3$ has no global maximum nor minimum at all. We can however look at global maxima on finite intervals.

Example: Let us look at the example from last week where we found the square of maximal area among all squares of side length $x, 1-x$. The function $f(x) = x(1-x)$ had a maximum at $x = 1/2$. We also have to look at the boundary points. Why? Because both x and $f(x)$ can not become negative. We see that $f(x)$ has to be looked at on the interval $[0, 1]$. We write $[0, 1]$ if we mean that 0 and 1 are included. To decide about global maxima, just look at the critical points and boundary points and pick the maximal.

Example: Find the global maximum of $f(x) = x^2$ on the interval $[-1, 2]$. **Solution**. We look for local extrema at critical points and at the boundary. Then we compare all these extrema to find the maximum or minimum. The critical points are $x = 0$. The boundary points are $-1, 2$. Comparing the values $f(-1) = 1, f(0) = 0$ and $f(4) = 4$ shows that f has a global maximum at 2 and a global minimum at 0.

> **Extreme value theorem of Bolzano**: A continuous function f on a closed finite interval $[a, b]$ attains a global maximum and a global minimum.

Proof: for every n, make a list of the points $x_k = (a+(k/n)(b-a))$ where $k = 1, \dots, n$. Pick the one where $f(x_k)$ is maximal one and call this x_n. Now we use the **Bolzano-Weierstrass theorem** which assures that any sequence of numbers x_n on a closed interval $[a, b]$ has an accumulation point. Such an accumulation point is a maximum. Similarly, we can construct the minimum.

The **Bolzano-Weierstrass theorem** is verified constructively too: cut the interval in two equal parts and choose a part which contains infinitely many points x_n. We have reduced the problem to a smaller interval. Now take this interval and again divide it into two. Relabel the points there with x_n. Again chose the one in which x_n hits infinitely many times. Washing, rinse and repeating this again and again leads to smaller and smaller intervals of size $[b-a]/2^n$ in which there are infinitely many points. Note that these intervals are nested so that they lead to a limit (if the interval were $[0,1]$ and we would cut each time into 10 pieces, then we would gain in every step one digit of the decimal expansion of the number we are looking for).

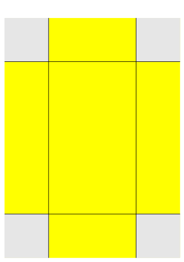
12.3. Note that the global maximum or minimum can also be on the boundary or points where the derivative does not exist:

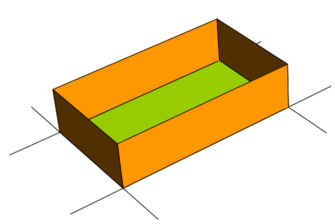
Example: Find the global maximum and minimum of the function $f(x) = |x|$. The function has no absolute maximum as it goes to infinity for $x \to \infty$. The function has a global minimum at $x = 0$ but the function is not differentiable there. The point $x = 0$ is a point which does not belong to the domain of f'.

Example: A **soda can** is a cylinder of volume $\pi r^2 h$. The surface area $2\pi r h + 2\pi r^2$ measures the amount of material used to manufacture the can. Assume the surface area is 2π, we can solve the equation for $h = (1-r^2)/r = 1/r - r$ **Solution:** The volume is $f(r) = \pi(r - r^3)$. Find the can with maximal volume: $f'(r) = \pi - 3r^2\pi = 0$ showing $r = 1/\sqrt{3}$. This leads to $h = 2/\sqrt{3}$.

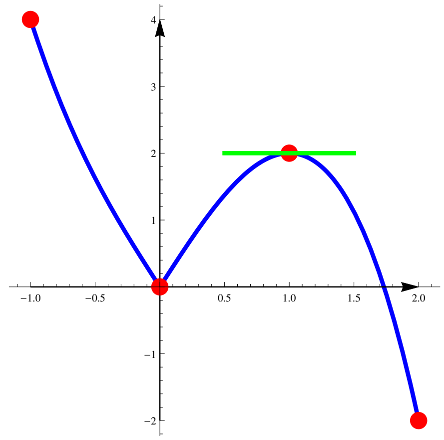
Example: Take a card of 2×2 inches. If we cut out 4 squares of equal side length x at the corners, we can fold up the paper to a tray with width $(2-2x)$ length $(2-2x)$ and height x. For which $x \in [0,1]$ is the tray volume maximal?

Solution The volume is $f(x) = (2-2x)(2-2x)x$. To find the maximum, we need to compare the critical points which is at $x = 1/3$ and the boundary points $x = 0$ and $x = 1$.





Example: Find the global maxima and minima of the function $f(x) = 3|x| - x^3$ on the interval $[-1, 2]$.
Solution. For $x > 0$ the function is $3x - x^3$ which can be differentiated. The derivative $3 - 3x^2$ is zero at $x = 1$. For $x < 0$ the function is $-3x - x^3$. The derivative is $-3 - x^2$ and has no root. The only critical points are 1. There is also the point $x = 0$ which is not in the domain where we can differentiate the function. We have to deal with this point separately. We also have to look at the boundary points $x = -1$ and $x = 2$. Making a list of function values at $x = -1, x = 0, x = 1, x = 2$ gives the maximum.



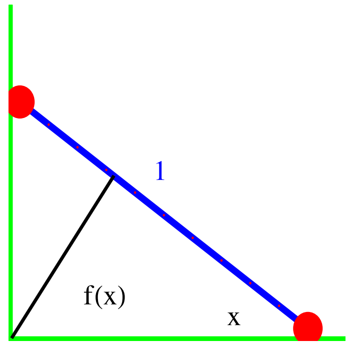
Homework

Problem 1: Find all the local maxima and minima as well as the global maximum and the global minimum of the function $f(x) = x^4 - 4x^3 - 2x^2 + 12x$ on the closed interval $[-4, 4]$. Make sure to compute the critical points inside the interval and then compare also the boundary points.

Problem 2: Find the global maximum and minimum of the function $f(x) = 2\sin(x) + x$ on the interval $[-10, 10]$.

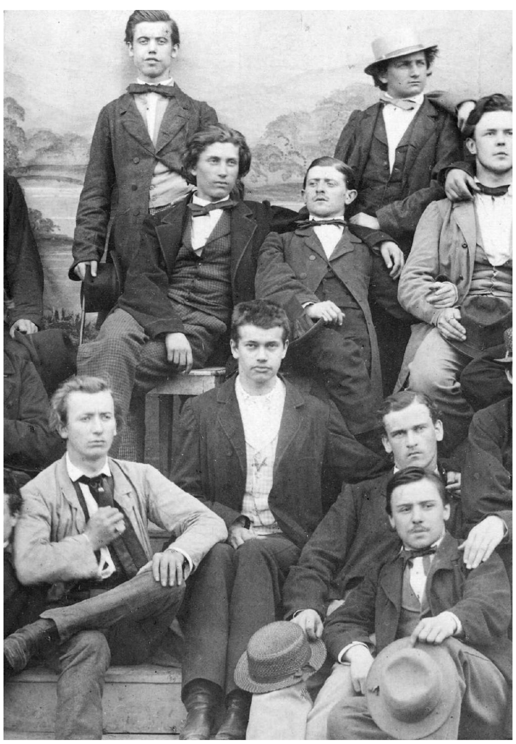
Problem 3: Mathcandy.com (look it up!) manufactures spherical candies. Its effectiveness is $A(r) - V(r)$, where $A(r)$ is the surface area and $V(r)$ the volume of a candy of radius r. Find the radius, where $f(r) = A(r) - V(r)$ has a global maximum for $r \geq 0$.

Problem 4: A ladder of length 1 is one side at a wall and on one side at the floor. a) Verify that the distance from the ladder to the corner is $f(x) = \sin(x)\cos(x)$.
b) Find the angle x for which $f(x)$ is maximal.



Problem 5: a) The function $S(x) = -x\log(x)$ is called the **entropy function**. Find the probability $0 < x \leq 1$ which maximizes entropy.
b) Find the global minimum of the **Helmholtz free energy** $G = H - TS$, where $T = 10$ is temperature, $S(x)$ is the entropy function in a) and $H = x$ is the **internal energy**.
P.S. One of the most important principles in science is that nature tries to maximize entropy or minimize fee energy.

Entropy has been introduced by Ludwig Boltzmann. It is important in physics and chemistry. $S = k\log(W)$ is interpreted as $W = 1/p$, then take the expectation giving $S = -k\sum_p p\log(p)$. Note the use of log and not ln. Hermann von Helmholz (1812-1894).


Boltzmann (1844-1906)


Helmholtz (1812-1894).

Unit 13: Hospital's rule

Lecture

13.1. Hospital's rule allows to compute limits. [1] It is a miracle procedure:

Hospital's rule. If f, g are differentiable and $f(p) = g(p) = 0$ and $g'(p) \neq 0$, then

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f'(x)}{g'(x)} .$$

Lets see how it works in examples:

Example: Lets prove **the fundamental theorem of trigonometry** again:

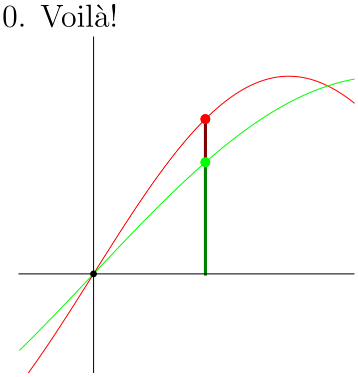
$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1 .$$

In order to apply l'Hospital, we had to know the derivative. Our work to establish the limit was not in vain.

13.2. The proof of the rule is very simple: since $f(p) = g(p) = 0$ we have $Df(p) = (f(p+h) - f(p))/h = f(p+h)/h$ and $Dg(p) = (g(p+h) - g(p))/h = g(p+h)/h$ so that for every $h > 0$ with $g(p+h) \neq 0$ the **quantum l'Hospital rule** holds:

$$\frac{f(p+h)}{g(p+h)} = \frac{Df(p)}{Dg(p)} .$$

Now take the limit $h \to 0$. Voilà!



Example: **Problem.** Find the limit $f(x) = (\exp(2x) - 1)/x$ for $x \to 0$. **Answer.** The rule gives 2.

[1]Also Hôpital. Hospital is is easier to write and remember (bring f to the hospital!)

13.3. Sometimes, we have to administer l'Hospital twice:

If $f(p) = g(p) = f'(p) = g'(p) = 0$ then $\lim_{x\to p} \frac{f(x)}{g(x)} = \lim_{x\to p} \frac{f''(x)}{g''(x)}$ if the limit to the right exists.

Example: **Problem.** Find the limit $f(x) = (\exp(x^2) - 1)/x^2$ for $x \to 0$.

Example: Find the limit $\lim_{x\to 0}(1 - \cos(x))/x^2$. This limit had been pivotal to compute the derivatives of trigonometric functions. **Solution**: differentiation gives

$$\lim_{x\to 0} - \sin(x)/2x \; .$$

Now apply l'Hospital again.

$$\lim_{x\to 0} - \sin(x)/(2x) = \lim_{x\to 0} - \cos(x)/2 = -\frac{1}{2} \; .$$

Example: **Problem:** What do you get if you apply l'Hospital to the limit $[f(x + h) - f(x)]/h$ as $h \to 0$?
Answer: Differentiate both sides with respect to h! And then feel awesome!

Example: Find $\lim_{x\to\infty} x\sin(1/x)$. **Solution.** Write $y = 1/x$ then $\sin(y)/y$. Now we have a limit, where the denominator and nominator both go to zero.

13.4. The case when both sides converge to infinity can be reduced to the 0/0 case by looking at $A = f/g = (1/g(x))/(1/f(x))$ which has the limit $g'(x)/g^2(x)/f'(x)/f^2(x) = g'(x)/f'(x)((1/g)/(1/f))^2 = g'/f'(f^2/g^2) = (g'/f')A^2$, so that $A = f'(p)/g'(p)$. We see:

If $\lim_{x\to p} f(x) = \lim_{x\to p} g(x) = \infty$ for $x \to p$ and $g'(p) \neq 0$, then

$$\lim_{x\to p} \frac{f(x)}{g(x)} = \frac{f'(p)}{g'(p)} \; .$$

Example: What is the limit $\lim_{x\to 0} x^x$? This will provide the best answer to the question **What is 0^0?**
Solution: Because $x^x = e^{x\log(x)}$, it is enough to understand the limit $x\log(x)$ for $x \to 0$.

$$\lim_{x\to 0} \frac{\log(x)}{1/x} \; .$$

Now the limit can be seen as the limit $(1/x)/(-1/x^2) = -x$ which goes to 0. Therefore $\lim_{x\to 0} x^x = 1$. (We assume that $x > 0$ in order to have real values x^x. If we want a function defined everywhere take $|x|^{|x|}$.)

Example: Find the limit $\lim_{x\to 2} \frac{x^2-4x+4}{\sin^2(x-2)}$.
Solution: this is a case where $f(2) = f'(2) = g(2) = g'(2) = 0$ but $g''(0) = 2$. The limit is $f''(2)/g''(2) = 2/2 = 1$.

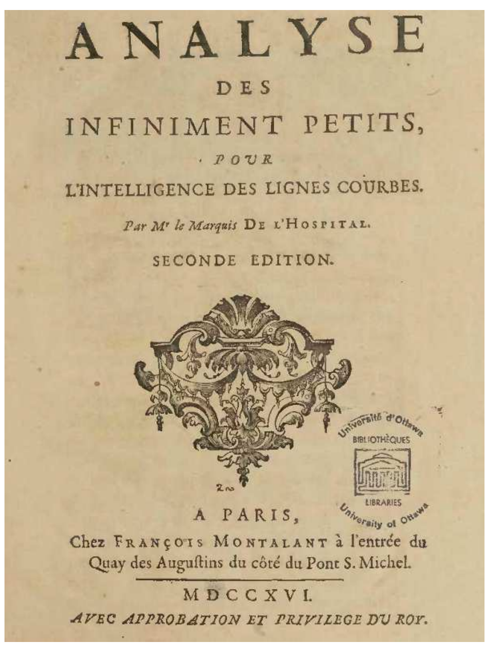
13.5. Hospital's rule always works in calculus situations, where functions are differentiable. The rule can fail if differentiability of f or g fails. Here is an other "rare" example, where one has to think a bit more:

Example: **Deja Vue**: Find $\frac{\sqrt{x^2+1}}{x}$ for $x \to \infty$. L'Hospital gives $x/\sqrt{x^2+1}$ which in terms gives again $\frac{\sqrt{x^2+1}}{x}$. Apply l'Hospital again to get the original function. We got an infinite loop. If the limit is A, then the procedure tells that it is equal to $1/A$. The limit must therefore be 1. This case can be covered easily without going to the hospital: divide both sides by x to get $\sqrt{1+1/x^2}$. Now, we can see the limit 1.

Example: **Trouble?** The limit $\lim_{x\to\infty}(2x+\sin(x))/3x$ is clearly $2/3$ since we can take the sum apart and have $2/3+\sin(x)/(3x)$. Hospital gives $\lim_{x\to\infty}(2+\cos(x))/3$ which has no limit. This is not trouble, since Hospital applies only if the limit $f'(x)$ and $g'(x)$ exists.

History

13.6. The "first calculus book", the world has known was "Analyse des Infiniment Petits pour l"intelligence des Lignes Courbes". It appeared in 1696 and was written by **Guillaume de l'Hospital**, a text if typeset in a modern font would probably fit onto 50-100 pages. [2] It is now clear that the mathematical content in Hospital's book is mostly due to **Johannes Bernoulli**. The book remained the standard for calculus textbooks for a century.


ANALYSE
DES
INFINIMENT PETITS,
POUR
L'INTELLIGENCE DES LIGNES COURBES.
Par Mr le Marquis DE L'HOSPITAL.
SECONDE EDITION.

A PARIS,
Chez FRANÇOIS MONTALANT à l'entrée du Quay des Auguſtins du côté du Pont S. Michel.
MDCCXVI.
AVEC APPROBATION ET PRIVILEGE DU ROY.

[2] Stewart's book with 1200 pages probably contains about 4 million characters, about 12 times more than l'Hospital's book. Modern calculus books also contain more material of course. The OCR text of l'Hospital's book of 200 pages has 300'000 characters.

Homework

Problem 13.1: For the following functions, find the limits as $x \to 0$:
a) $\sin(7x)/5x$
b) $(\exp(16x) - 1)/(\exp(17x) - 1)$
c) $\sin^2(8x)/\sin^2(5x)$
d) $\frac{\tan(4x)}{3x}$
e) $\sin(\sin(11x))/x$.

Problem 13.2: Find the following limits which are indefinite forms ∞/∞ a) $\lim_{x\to 0} \cot(x)/\cot(3x)$.
b) $\lim_{x\to\infty} \frac{3x^2+1}{4x^2+100}$.
c) Find $\lim_{x\to\infty}(x^2 + x - 1)/\sqrt{5x^4 + 1}$.
(**Hint.** To compute the limit faster, Find the limit of $(x^2+x-1)^2/(9x^4+1)$ first, then take the square root of the limit. Apply Hospital several times).

Problem 13.3: Use l'Hospital to compute the following limits $x \to 0$:
a) $\lim_{x\to 0} x/\log|x|$
b) $\log|5x|/\log|x|$.
c) $4\text{sinc}'(x) = 4(\cos(x)x - \sin(x))/x^2$
d) $\log|1 + x|/\log|\log|1 + x||$.
e) $(e^x - 1)/(e^{2x} - 1)$

Problem 13.4: We have seen how to compute limits with healing. Solve this now with l'Hospital at $x \to 1$:
a) $\frac{x^{100}-1}{x^{22}-1}$.
b) $\frac{\tan^2(x-1)}{(\cos(x-1)-1)}$

Problem 13.5: More practice.
a) Find the limit $\lim_{x\to 0} \frac{x}{\tan(6x)}$.
b) Find the limit $\lim_{x\to 5} \frac{x^2-25}{x-5}$
c) Find the limit $\lim_{x\to 0} \frac{1-e^x}{x-x^3}$.
d) Find the limit $\lim_{x\to 0} \frac{\log(1+9x)}{4x}$.
e) Find the limit $\lim_{x\to 1}(x^7 - 1)/(x^3 - 1)$.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 14: Newton method

Lecture

14.1. We were able to find roots of functions using a "divide and conquer" technique: start with an interval $[a, b]$ for which $f(a) < 0$ and $f(b) > 0$. If $f((a+b)/2)$ is positive, then use the interval $[a, (a+b)/2]$ otherwise $[(a+b)/2, b]$. After n steps, we are $(b-a)/2^n$ close to the root. If the function f is differentiable, we can do better and use the value of the derivative to get closer to a point $y = T(x)$. Lets find this point y. If we draw a tangent at $(x, f(x))$ and intersect it with the x-axes, then

$$f'(x) = \frac{f(x) - 0}{x - T(x)} .$$

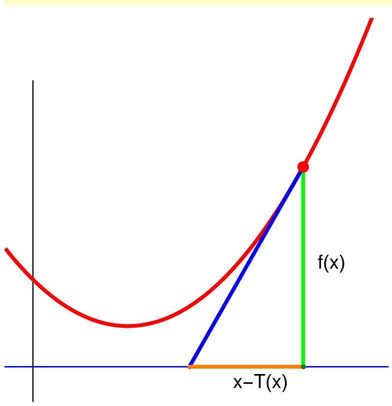
Now, $f'(x)$ is the slope of the tangent and the right hand side is "rise" over "run" (see the picture). If we solve for $T(x)$, we get

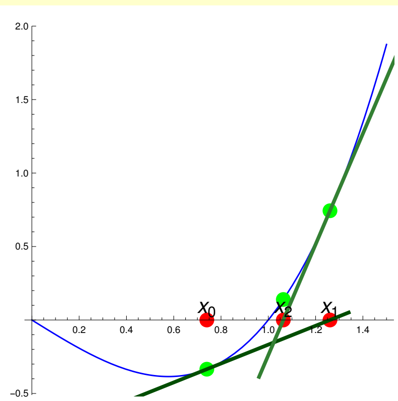
Definition: The **Newton map** is defined as

$$T(x) = x - \frac{f(x)}{f'(x)} .$$

14.2. The **Newton's method** applies this map a couple of times until we are sufficiently close to the root: start with a point x, then compute a new point $x_1 = T(x)$, then $x_2 = T(x_1)$ etc.

If p is a root of f such that $f'(p) \neq 0$, and x_0 is close enough to p, then $x_1 = T(x), x_2 = T^2(x)$ converges to the root p.





Example: If $f(x) = ax + b$, we reach the root in one step.

Example: If $f(x) = x^2$ then $T(x) = x - x^2/(2x) = x/2$. We get exponentially fast to the root 0. In general, the method is much better:

The Newton method converges extremely fast to a root $f(p) = 0$ if $f'(p) \neq 0$. In general, the number of correct digits double in each step.

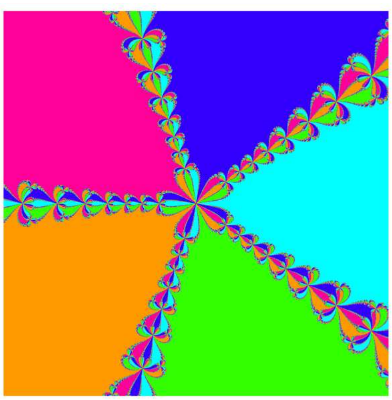
In 4 steps we expect to have $2^4 = 16$ digits correct. Having a fast method to compute roots is useful. For example, in computer graphics, where things can not be fast enough. We will explore a bit in the lecture how fast the method is.

14.3. If we have several roots, and we start at some point, to which root will the Newton method converge? Does it at all converge? This is an interesting question. It is also historically intriguing because it is one of the first cases, where "chaos" was observed at the end of the 19'th century.

Example: Find the Newton map in the case $f(x) = x^5 - 1$.
Solution

$$T(x) = x - \frac{x^5 - 1}{5x^4} .$$

14.4. If we look for roots in the complex like for $f(x) = x^5 - 1$ which has 5 roots in the complex plane, the "region of attraction" of each of the roots is a complicated set which we call the **Newton fractal**.



The Newton method is scrumtrulescent! (To quote from the "Inside the Actors Studio" at SNL)

Example: Lets compute $\sqrt{2}$ to 12 digits accuracy. We want to find a root $f(x) = x^2 - 2$. The Newton map is $T(x) = x - (x^2 - 2)/(2x)$. Lets start with $x = 1$.

$$\begin{aligned} T(1) &= 1 - (1-2)/2 = 3/2 \\ T(3/2) &= 3/2 - ((3/2)^2 - 2)/3 = 17/12 \\ T(17/12) &= 577/408 \\ T(577/408) &= 665857/470832 . \end{aligned}$$

This is already $1.6 \cdot 10^{-12}$ close to the real root! 12 digits, by hand!

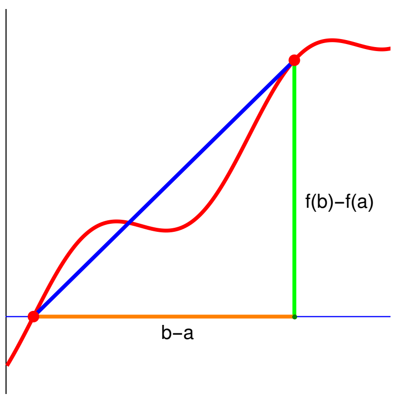
Example: To find the cube root of 10 we have to find a root of $f(x) = x^3 - 10$. The Newton map is $T(x) = x - (x^3 - 10)/(3x^2)$. If we start with $x = 2$, we get the following steps $2, 13/6, 3277/1521, 105569067476/49000820427$. After three steps we have a result which is already $2.2 \cdot 10^{-9}$ close to the root.

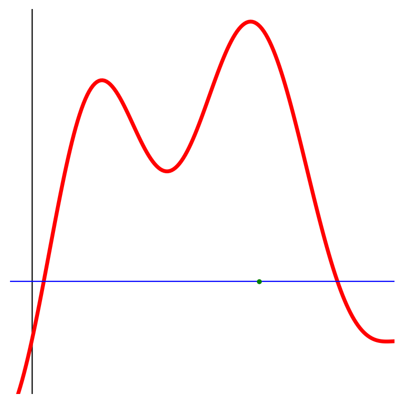
Example: Verify that the Newton map $T(x)$ in the case $f(x) = a(x-b)^n$ with $n > 0$ has the property that for the root $x = b$ is obtained.

14.5. Unlike the intermediate value theorem which applied for continuous functions, the mean value theorem involves derivatives. We assume therefore today that all functions are differentiable unless specified. The mean value theorem can be seen as a consequence of the intermediate value theorem.

Mean value theorem: Any interval (a, b) contains a point x such that

$$f'(x) = \frac{f(b) - f(a)}{b - a} .$$





Here are a few examples which illustrate the theorem:

Example: Verify with the mean value theorem that the function $f(x) = x^2 + 4\sin(\pi x) + 5$ has a point where the derivative is 1.

Solution. Since $f(0) = 5$ and $f(1) = 6$ we see that $(f(1) - f(0))/(1 - 0) = 5$.

Example: A biker drives with velocity $f'(t)$ at position $f(b)$ at time b and at position a at time a. The value $f(b)-f(a)$ is the distance traveled. The fraction $[f(b)-f(a)]/(b-a)$ is the average speed. The theorem tells that there was a time when the bike had exactly the average speed.

14.6. Proof of the theorem: the function $h(x) = f(a)+cx$, where $c = (f(b)-f(a)/(b-a)$ also connects the beginning and end point. The function $g(x) = f(x) - h(x)$ has now the property that $g(a) = g(b)$. If we can show that for such a function, there exists x with $g'(x) = 0$, then we are done. By tilting the picture, we have reduced it to a statement seen before:

Rolle's theorem: If $f(a) = f(b)$ then f has a critical point in (a, b).

Proof: If it were not true, then either $f'(x) > 0$ everywhere implying $f(b) > f(a)$ or $f'(x) < 0$ implying $f(b) < f(a)$.

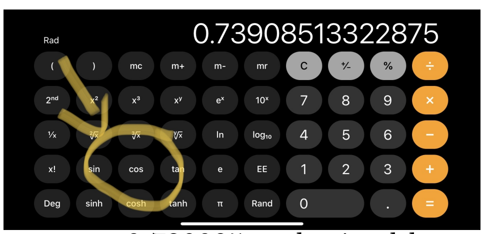
Example: Show that the function $f(x) = \sin(x) + x(\pi - x)$ has a critical point $[0, \pi]$.
Solution: The function is differentiable and non-negative. It is zero at $0, \pi$. By Rolle's theorem, there is a critical point.

Example: Verify that the function $f(x) = 2x^3 + 3x^2 + 6x + 1$ has only one real root. **Solution:** There is at least one real root by the intermediate value theorem: $f(-1) = -4, f(1) = 12$. Assume there would be two roots. Then by Rolle's theorem there would be a value x where $g(x) = f'(x) = 6x^2 + 6x + 6 = 0$. But there is no root of g. [The graph of g minimum at $g'(x) = 6 + 12x = 0$ which is 1/2 where $g(1/2) = 21/2 > 0$.]

Homework

Problem 14.1: Get the Newton map $T(x) = x - f(x)/f'(x)$ for:
a) $f(x) = (x-2)^2$
b) $f(x) = e^{5x}$
c) $f(x) = 2e^{-x^2}$
d) $f(x) = \cot(x)$.

Problem 14.2: The function $f(x) = \cos(x) - x$ has a root between 0 and 1. Starting with $x = 1$, perform the first Newton step



Compare with the root $x = 0.739085...$ obtained by punching "cos" again and again

Problem 14.3: We want to find the square root of 102. We have to solve $\sqrt{102} = x$ or $f(x) = x^2 - 102 = 0$. Perform a Newton step starting at $x = 10$.

Problem 14.4: Find the Newton step $T(x) = x - f(x)/f'(x)$ in the case $f(x) = 1/x$. What happens if you apply the Newton steps starting with $x = 1$? Does the method converge?

Problem 14.5: We look at the function $f(x) = x^{10} + x^4 - 20x$ on the positive real line. Use the **mean value theorem** on $(1, 2)$ to assure the there exists x, where $g(x) = f'(x) - [f(2) - f(1)] = f'(x) - 1018$. Now use a single Newton step starting with 1.5 to find an approximate solution to $g(x) = 0$.

Unit 15: Review

Major points

f is **continuous** at a if there is $b = f(a)$ such that $\lim_{x \to a} f(x) = b$ for every a. The intermediate value theorem: $f(a) > 0, f(b) < 0$ implies f having a root in (a, b).

$f'(x) = 0, f''(x) > 0$ then x is **local min**. $f'(x) = 0,'' (x) < 0$ then x is **local max**. For **global minima or maxima**, compare local extrema and boundary values.

If f changes sign we have a **root** $f = 0$, if f' changes sign, we have a **critical point** $f' = 0$ if f'' changes sign, we have an **inflection points**. A function is **even** if $f(-x) = f(x)$, and **odd** if $f(-x) = -f(x)$.

If $f' > 0$ then f is increasing, if $f' < 0$ it is decreasing. If $f''(x) > 0$ it is **concave up**, if $f''(x) < 0$ it is **concave down**. If $f'(x) = 0$ then f has a horizontal tangent.

Hosptal's theorem applies for indeterminate forms $0/0$ or ∞/∞. In that case, $\lim_{x \to a} f(x)/g(x)$, where $f(a) = g(a) = 0$ or $f(a) = g(a) = \infty$ with $g'(a) \neq 0$ are given by $f'(a)/g'(a)$.

With $Df(x) = (f(x+h) - f(x))/h$ and $S(x) = h(f(0) + f(2h) + \cdots + f((k-1)h))$ we have a **preliminary fundamental theorem of calculus** $SDf(kh) = f(kh) - f(0)$ and $DS(f(kh)) = f(kh)$.

Roots of $f(x)$ with $f(a) < 0, f(b) > 0$ can be obtained by the dissection method by applying the **Newton map** $T(x) = x - f(x)/f'(x)$ again and again.

Algebra reminders

Healing:	$(a+b)(a-b) = a^2 - b^2$ or $1 + a + a^2 + a^3 + a^4 = (a^5 - 1)/(a - 1)$
Denominator:	$1/a + 1/b = (a+b)/(ab)$
Exponential:	$(e^a)^b = e^{ab}$, $e^a e^b = e^{a+b}$, $a^b = e^{b \log(a)}$
Logarithm:	$\log(ab) = \log(a) + \log(b)$. $\log(a^b) = b \log(a)$
Trig functions:	$\cos^2(x) + \sin^2(x) = 1, \sin(2x) = 2\sin(x)\cos(x), \cos(2x) = \cos^2(x) - \sin^2(x)$
Square roots:	$a^{1/2} = \sqrt{a}$, $a^{-1/2} = 1/\sqrt{a}$

Important functions

Polynomials	$x^3 + 2x^2 + 3x + 1$	Exponential	$5e^{3x}$
Rational functions	$(x + 1)/(x^3 + 2x + 1)$	Logarithm	$\log(3x)$
Trig functions	$2\cos(3x)$	Inverse trig functions	$\arctan(x)$

Important derivatives

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$f(x) = x^n$	nx^{n-1}	$f(x) = \sin(ax)$	$a\cos(ax)$
$f(x) = e^{ax}$	ae^{ax}	$f(x) = \tan(x)$	$1/\cos^2(x)$
$f(x) = \cos(ax)$	$-a\sin(ax)$	$f(x) = \log(x)$	$1/x$
$f(x) = \arctan(x)$	$1/(1 + x^2)$	$f(x) = \sqrt{x}$	$1/(2\sqrt{x})$

Differentiation rules

Addition rule	$(f + g)' = f' + g'$.	Quotient rule	$(f/g)' = (f'g - fg')/g^2$.
Scaling rule	$(cf)' = cf'$.	Chain rule	$(f(g(x))' = f'(g(x))g'(x)$.
Product rule	$(fg)' = f'g + fg'$.	Easy rule	simplify before deriving

Extremal problems

To maximize or minimze f on an interval $[a, b]$, find all critical points inside the interval, evaluate f on the boundary $f(a), f(b)$ and then compare the values to find the global maximum. No second derivative test at the boundary.

Limit examples

$\lim_{x\to 0} \sin(x)/x$	l'Hospital 0/0	$\lim_{x\to 1}(x^2 - 1)/(x - 1)$	heal
$\lim_{x\to 0}(1 - \cos(x))/x^2$	l'Hospital 0/0 twice	$\lim_{x\to\infty} \exp(x)/(1 + \exp(x))$	l'Hospital
$\lim_{x\to 0}(1/x)/\log(x)$	l'Hospital ∞/∞	$\lim_{x\to 0}(x + 1)/(x + 5)$	no work necessary

Important things

Summation and rate of change are at the heart of calculus.
The 3 major types of discontinuities are jump, oscillation, infinity.
Dissection and Newton methods are algorithms to find roots.
The fundamental theorem of trigonometry is $\lim_{x\to 0} \sin(x)/x = 1$.
The derivative is the limit $Df(x) = [f(x + h) - f(x)]/h$ as $h \to 0$.
The rule $D(1 + h)^{x/h} = (1 + h)^{x/h}$ leads to $\exp'(x) = \exp(x)$.
If you forget a derivative like of $\arctan(x)$, use the chain rule.

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INTRODUCTION TO CALCULUS

MATH 1A

Unit 16: Catastrophes

Lecture

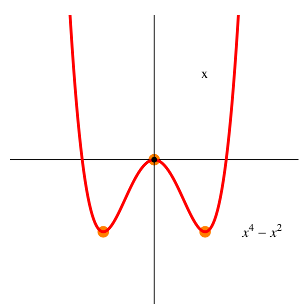
16.1. In this lecture, we are interested how minima and maxima change when a parameter is changed. Nature, economies, or processes likes extrema. It turns out that we change parameters, the outcome changes often in a non-smooth way. An economic parameter can change quickly for example. One calls this a catastrophe. This can be explained with mathematics. A key are **stable equilibria**, local minimum. Here is a general principle:

If a local minimum disappears when we change an external parameter, the system settles in a new stable equilibrium. The new equilibrium can be far away from the original one.

16.2. To see this, let us look at the following optimization problem

Example: Find all the minima and maxima of the function

$$f(x) = x^4 - x^2$$

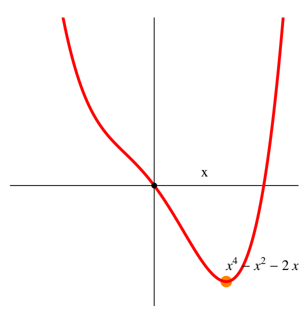


Solution: $f'(x) = 4x^3 - 2x$ is zero for $x = 0, 1/\sqrt{2}, -1/\sqrt{2}$. The second derivative is $12x^2 - 2$. It is negative for $x = 0$ and positive at the other two points. We have two local minima and one local maximum.

Example: Now find all the extrema of the function

$$f(x) = x^4 - x^2 - 2x$$

There is only one critical point. It is $x = 1$.

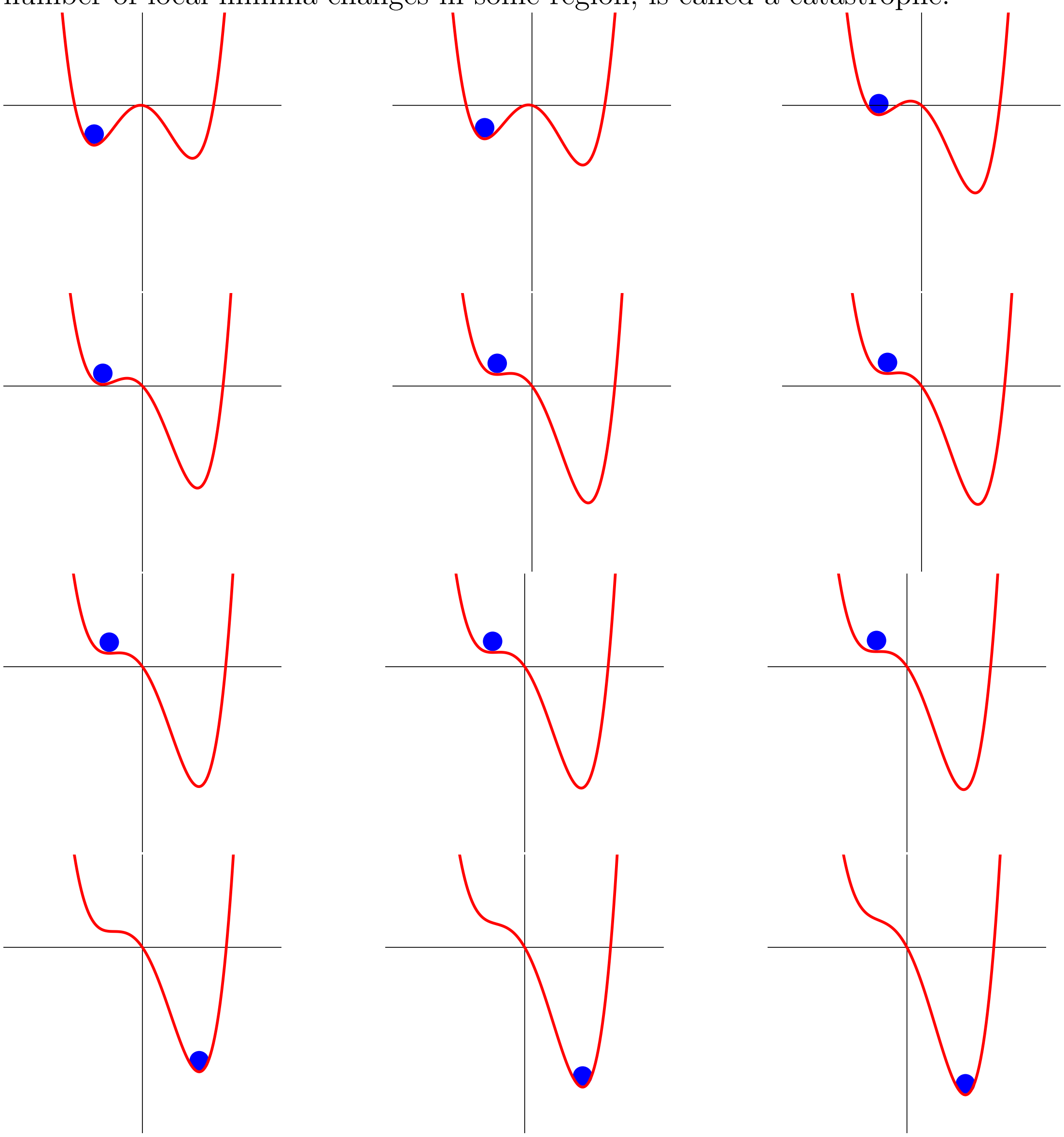


16.3. When the first graph is morphed into the second example, the local minimum to the left has disappeared. Assume the function f measures the prosperity of some kind and c is a **parameter**. We look at the position of the first critical point of the function. Catastrophe theorists look at the following **assumption**:

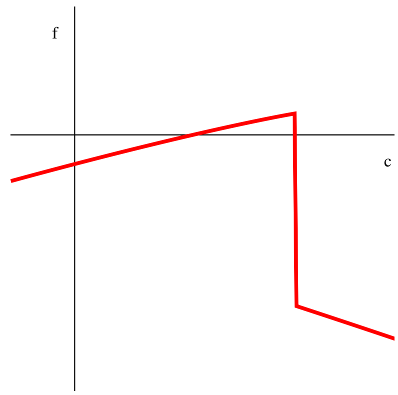
Definition: Assume the system depends on a parameter, the minimum, **stable equilibrium** depends on this parameter. It remains a stable equilibrium until it disappears. If that happens, the system settles in a neighboring stable equilibrium.

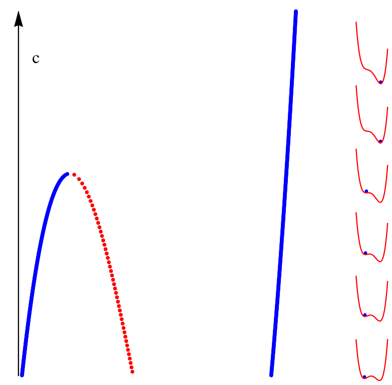
Definition: A parameter value c_0 at which somewhere a stable minimum disappears or appears, is called a **catastrophe**. In other words, if for $c < c_0$ a different collection of local minima exist than for $c > c_0$, then the parameter value c_0 is called a **catastrophe**.

16.4. In order to visualize a catastrophe, we draw the graphs of the function $f_c(x)$ for various parameters c and look at the local minima. At a parameter value, where the number of local minima changes in some region, is called a catastrophe.



16.5. You see that in this particular case, the catastrophe has happened between the 9'th and 10'th picture. Here is the position of the equilibrium point in dependence of c.





16.6. A **bifurcation diagram** displays the equilibrium points as they change in dependence of the parameter c. The vertical axes is the parameter c, the horizontal axes is x. At the bottom for $c = 0$, there are three equilibrium points, two local minima and one local maximum. At the top for $c = 1$ we have only one local minimum. Here is an important principle:

Catastrophes often lead to a strict and abrupt decrease of the minimal critical value. It is not possible to reverse the process in general.

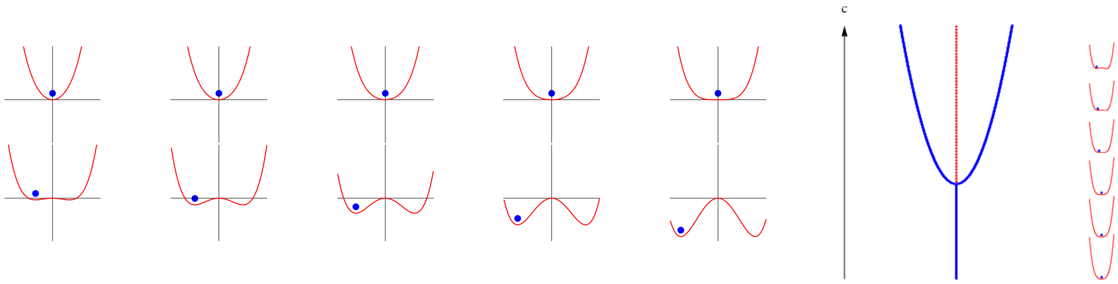
16.7. Let us look at this "movie" of graphs and run it backwards. By the same principle we do not end up at the position we started with. The new equilibrium remains the equilibrium nearby.

Catastrophes are in general **irreversible**.

16.8. We know this from experience: it is easy to screw up a relationship, get sick, have a ligament torn or lose somebody's trust. Building up a relationship, getting healthy or gaining trust usually happens continuously and slowly. Ruining the economy of a country or a company or losing a good reputation of a brand can be quick. It takes time to regain it.

Local minima can change discontinuously, when a parameter is changed. This can happen with perfectly smooth functions and smooth parameter changes.

Example: We look at the example $f(x) = x^4 - cx^2$ with $-1 \leq c \leq 1$ in class.



Homework

In this homework, we study a catastrophe for the function

$$f(x) = x^6 - x^4 + cx^2 ,$$

where c is a parameter between 0 and 1.

Problem 16.1: a) Find all the critical points in the case $c = 0$ and analyze their stability. b) Find all the critical points in the case $c = 1$ and analyze their stability.

Problem 16.2: Plot the graph of the function $f(x)$ for 10 values of c between 0 and 1. You can use desmos or Wolfram alpha. Mathematica example code is below.

Problem 16.3: If you change from $c = -0.3$ to 0.6, pinpoint the c value for the catastrophe and show a rough plot of $c \to f(x_c)$, the value at the first local minimum x_c in dependence of c. The text above provides this graph for an other function. It is the graph with a discontinuity.

Problem 16.4: If you change back from $c = 0.6$ to -0.3 pinpoint the value for the catastrophe. It will be different from the one in the previous question.

Problem 16.5: Sketch the bifurcation diagram. That is, if $x_k(c)$ is the k'th equilibrium point, then draw the union of all graphs of $x_k(c)$ as a function of c (the c-axes pointing upwards). As in the two example provided, draw the local maximum with dotted lines.

```
Manipulate[ Plot[x^6 - x^4 + c x^2, {x, -1, 1}], {c, 0, 1}]
```

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Unit 17: Riemann Integral

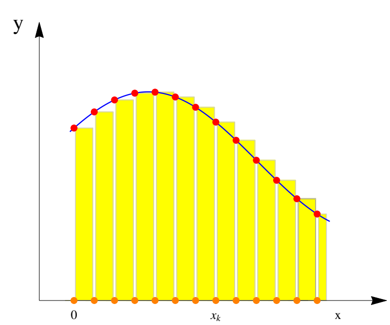
Lecture

17.1. In this lecture, we define the definite integral $\int_0^x f(t)\,dt$ if f is a differentiable function. We then compute it for some basic functions. We have previously defined the **Riemann sums**

$$Sf(x) = h[\ f(0) + f(h) + f(2h) + \cdots + f(kh)]\ ,$$

where k is the largest integer such that $kh < x$. Lets write S_n if we want to stress that the parameter $h = 1/n$ was used in the sum. We define the **Riemann integral** as the limit of these sums $S_n f$, when the **mesh size** $h = 1/n$ goes to zero.

Definition: Define
$$\int_0^x f(t)\,dt = \lim_{n\to 0} S_n f(x)\ .$$



17.2. A very important result is that

For any continuous function, the limit exists.

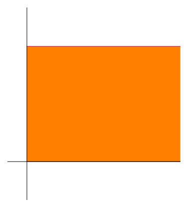
It is easier to see when f is f is differentiable as one can then estimate the error. There are n little pieces which are each of area $\leq M/n$, where M is the maximal slope that f can have in the given interval.

For non-negative f, the value $\int_0^x f(x)\,dx$ is the **area between the x-axis and the graph** of f. For general f, it is a **signed area**, the difference between two areas.

17.3. The Riemann integral is the limit $h\sum_{x_k=kh\in[0,x)} f(x_k)$. It converges to the area under the curve for all **continuous** functions. In probability theory, one uses also an other integral, the **Lebesgue integral**. It can be defined as the limit $\frac{1}{n}\sum_{k=1}^{n} f(x_k)$ where x_k are **random points** in $[0,x]$. This is a **Monte-Carlo integral** definition of the Lebesgue integral.

17.4. Riemann also looked also at points $x_0 < x_1 < \cdots < x_n$ $[0,x]$ such that the maximal distance $(x_{k+1} - x_k)$ between neighboring x_j goes to zero. The Riemann sum is then $S_n f = \sum_k f(y_k)(x_{k+1} - x_k)$, where y_k is arbitrarily chosen inside the interval (x_k, x_{k+1}). For continuous functions, the limiting result is the same the $Sf(x)$ sum done here. There are numerical reasons to allow more general partitions because it allows to adapt the mesh size: use more points where the function is complicated.

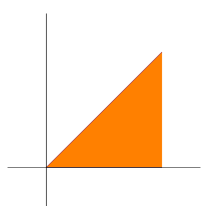
Example: If $f(x) = c$ is constant, then $\int_0^x f(t)\ dt = cx$. We can see also that $cnx/n \leq S_n f(x) \leq c(n+1)x/n$.



Example: Let $f(x) = cx$. The area is half of a rectangle of width x and height cx so that the area is $cx^2/2$. Adding up the Riemann sum is more difficult. Let k be the largest integer smaller than $xn = x/h$. Then

$$S_n f(x) = \frac{1}{n}\sum_{j=1}^{k} \frac{cj}{n} = \frac{ck(k+1)/2}{n^2} .$$

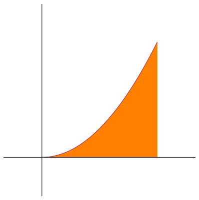
Taking the limit $n \to \infty$ and using that $k/n \to x$ shows that $\int_0^x f(t)\ dt = cx^2/2$.

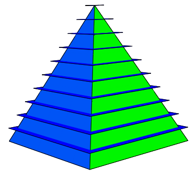


Example: Let $f(x) = x^2$. In this case,we can not see the numerical value of the area geometrically. But since we have computed $S[x^2]$ in the first lecture of this course and seen that it is $[x^3]/3$ and since we have defined $S_h f(x) \to \int_0^x f(t)\ dt$ for $h \to 0$ and $[x^k] \to x^k$ for $h \to 0$, we know that

$$\int_0^x t^2\ dt = \frac{x^3}{3} .$$

This example actually computes the **volume of a pyramid** which has at distance t from the top an area t^2 cross section. Think about $t^2 dt$ as a slice of the pyramid of area t^2 and height dt. Adding up the volumes of all these slices gives the volume.

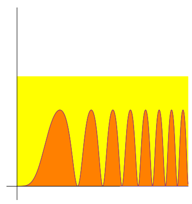




Linearity of the integral (see homework) $\int_0^x f(t)+g(t)\ dt = \int_0^t f(t)\ dt + \int_0^x g(t)\ dt$ and $\int_0^x \lambda f(t)\ dt = \lambda \int_0^x f(t)\ dt$.

Upper bound: If $0 \leq f(x) \leq M$ for all x, then $\int_0^x f(t)\ dt \leq Mx$.

Example: $\int_0^x \sin^2(\sin(\sin(t)))/x\ dt \leq x$. **Solution**. The function $f(t)$ inside the interval is non-negative and smaller or equal to 1 The graph of f is therefore contained in a rectangle of width x and height 1.



17.5. We see that if two functions are close then their difference is a function which is included in a small rectangle and therefore has a small integral:

If f and g satisfy $|f(x) - g(x)| \leq c$, then

$$\int_0^x |f(x) - g(x)|\ dx \leq cx\ .$$

17.6. We know identities like $S_n[x]_h^n = \frac{[x]_h^{n+1}}{n+1}$ and $S_n \exp_h(x) = \exp_h(x)$ already. Since $[x]_h^k - [x]^k \to 0$ we have $S_n[x]_h^k - S_n[x]^k \to 0$ and from $S_n[x]_h^k = [x]_h^{k+1}/(k+1)$. The other equalities are the same since $\exp_h(x) = \exp(x) \to 0$. This gives us:

$\int_0^x t^n\ dt = \frac{x^{n+1}}{n+1}$

$\int_0^x e^t\ dt = e^x - 1$

$\int_0^x \cos(t)\ dt = \sin(x)$

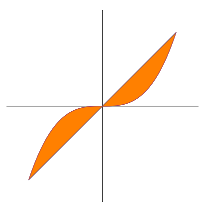
$\int_0^x \sin(t)\ dt = 1 - \cos(x)$

Homework

In the following homework you can use that $\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$ if F is a function which satisfies $F'(x) = f(x)$. We have already verified the identity for sums.

Problem 17.1: a) What is the integral $\int_0^1 4x^{24}\,dx$?
b) Find the integral $\int_0^{\log(2)} 5e^t\,dt$.
c) Calculate $\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx$.
d) Find $\int_0^{\pi/2} \cos^2(t)\,dt$.
e) Find $\int_0^{\pi/2} \sin^4(t)\,dt$.

Problem 17.2: The region enclosed by the graph of x and the graph of x^5 has a propeller type shape. Find its (positive) area.



Problem 17.3: Make a geometric picture for each of the following statements (which are rules for integration):

- $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$.
- $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$.
- $\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx$.

Problem 17.4: Here are some more challenging integrals. Maybe you have to guess or remember some old computations, maybe even look at the exam. a) $\int_0^2 x^x(1+\log(x))\,dx$
b) $\int_0^1 (3/2)\sqrt{1+x}\,dx$
c) $\int_0^{\sqrt{\log(2)}} 4xe^{-x^2}\,dx$
d) $\int_1^e 5\log(x)/x\,dx$

Problem 17.5: In this problem, it is crucial that you plot the function first. Split the integral up into parts. Find $\int_{-1}^4 f(x)\,dx$ for $f(x) = |x - |x-2||$.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 18: Fundamental theorem

Lecture

18.1. The **fundamental theorem of calculus** for differentiable functions will allow us to compute many integrals nicely. You have already made use of this theorem in the homework for today. Earlier in the course, we saw that $Sf(x) = h(f(0) + \cdots + f(kh))$ and $Df(x) = (f(x+h) - f(x))/h$ we have $SDf = f(x) - f(0)$ and $DSf(x) = f(x)$ if $x = nh$. This now becomes the **fundamental theorem**. It assumes that f' must be continuous.

$$\int_0^x f'(t)\, dt = f(x) - f(0) \text{ and } \frac{d}{dx} \int_0^x f(t)\, dt = f(x)$$

Proof. Using notation of Euler, we write $A \sim B$. We say "A and B are close" and mean that $A - B \to 0$ for $h \to 0$. [1] From $DSf(x) = f(x)$ for $x = kh$ we have $DSf(x) \sim f(x)$ for $kh < x < (k+1)h$ because f is continuous. We also know $\int_0^x Df(t)\, dt \sim \int_0^x f'(t)\, dt$ because $Df(t) \sim f'(t)$ uniformly for all $0 \leq t \leq x$ by the definition of the derivative and the assumption that f' is continuous and using Bolzano on the bounded interval. We also know $SDf(x) = f(x) - f(0)$ for $x = kh$. By definition of the Riemann integral, $Sf(x) \sim \int_0^x f(t)\, dt$ and so $SDf(x) \sim \int_0^x Df(t)\, dt$.

$$f(x) - f(0) \sim SDf(x) \sim \int_0^x Df(t)\, dt \sim \int_0^x f'(t)\, dt$$

as well as

$$f(x) \sim DSf(x) \sim D \int_0^x f(t)\, dt \sim \frac{d}{dx} \int_0^x f(t)\, dt\ .$$

Example: $\int_0^5 x^7\, dx = \frac{x^8}{8}\big|_0^5 = \frac{5^8}{8}$. You can always leave such expressions as your final result. It is even more elegant than the actual number 390625/8.

Example: $\int_0^{\pi/2} \cos(x)\, dx = \sin(x)\big|_0^{\pi/2} = 1$.

Example: Find $\int_0^\pi \sin(x)\, dx$. **Solution**: The answer is 2.

Example: For $\int_0^2 \cos(t+1)\, dt = \sin(x+1)\big|_0^2 = \sin(2) - \sin(1)$, the additional term $+1$ does not make matter as when using the chain rule, it goes away.

[1] Bolzano or Weierstrass would write $A \sim B$ as $\forall \epsilon > 0, \exists \delta > 0, |h| < \delta \Rightarrow |A - B| < \epsilon$ but who can parse this?

Example: For $\int_{\pi/6}^{\pi/4} \cot(x)\ dx$, the anti-derivative is difficult to spot. It becomes only accessible if we know, where to look: the function $\log(\sin(x))$ has the derivative $\cos(x)/\sin(x)$. So, we know the answer is $\log(\sin(x))|_{\pi/6}^{\pi/4} = \log(\sin(\pi/4)) - \log(\sin(\pi/6)) = \log(1/\sqrt{2}) - \log(1/2) = -\log(2)/2 + \log(2) = \log(2)/2$.
Let us look at two for now more challenging cases:
Example: The example $\int_2^3 2/(t^2-1)\ dt$ is challenging for now. We need a hint and write $2/(x^2-1) = 1/(x-1) - 1/(x+1)$. The function $F(x) = \log|x-1| - \log|x+1|$ has therefore $f(x) = 2/(x^2-1)$ as a derivative. The answer is $\int_2^3 2/(t^2-1)\ dt = F(3) - F(2) = \log(2) - \log(4) - \log(1) + \log(3) = \log(3) - \log(2) = \log(3/2)$.
Example: $\int_0^x \cos(\sin(x))\cos(x)\ dx = \sin(\sin(x))$ because the derivative of $\sin(\sin(x))$ is $\cos(\sin(x))\cos(x)$. The function $\sin(\sin(x))$ is an **anti-derivative** of f. If we differentiate this function, we get $\cos(\sin(x))\cos(x)$. Also this can be hard to spot for now. We will learn how to do this
We give reformulations of the fundamental theorem in ways in which it is mostly used: If f is the derivative of a function F then

$$\int_a^b f(x)\ dx = F(x)|_a^b = F(b) - F(a)\ .$$

In some textbooks, this is called the "second fundamental theorem" or the "evaluation part" of the fundamental theorem of calculus. The statement $\frac{d}{dx}\int_0^x f(t)\ dt = f(x)$ is the "anti-derivative part" of the fundamental theorem. They obviously belong together and are two different sides of the same coin.

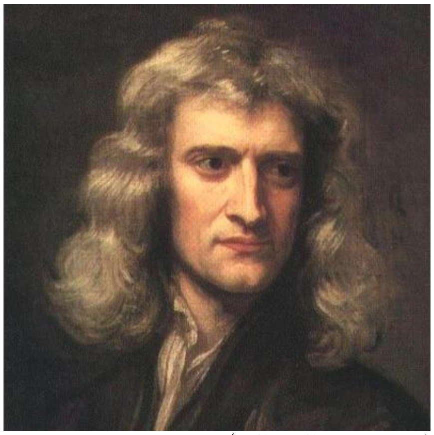
Here is a version of the fundamental theorem, where the boundaries are functions of x. Given functions g, h and if F is a function such that $F' = f$, then

$$\int_{h(x)}^{g(x)} f(t)\ dt = F(g(x)) - F(h(x))\ .$$

Example: $\int_{x^4}^{x^2} \cos(t)\ dt = \sin(x^2) - \sin(x^4)$.
The function F is called an **anti-derivative**. It is not unique but the above formula does always give the right result. Lets make a list You should have as many **anti-derivatives** "hard wired" in your brain. It really helps. Here are the core functions you should know.

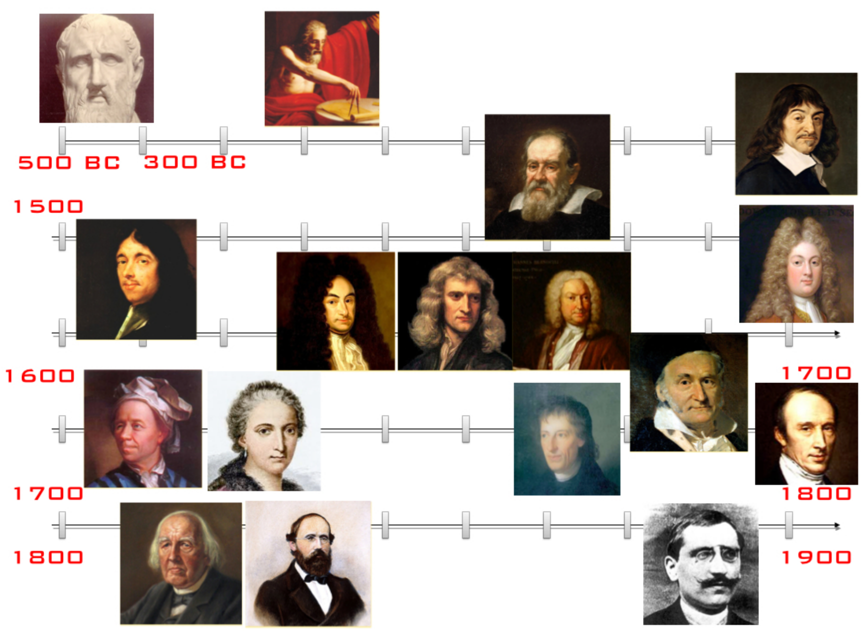
function	anti derivative
x^n	$\frac{x^{n+1}}{n+1}$
$\sqrt{x}$	$\frac{x^{3/2}2}{3}$
e^{ax}	$\frac{e^{ax}}{a}$
$\cos(ax)$	$\frac{\sin(ax)}{a}$
$\sin(ax)$	$\frac{-\cos(ax)}{a}$
$\frac{1}{x}$	$\log(x)$
$\frac{1}{1+x^2}$	$\arctan(x)$
$\log(x)$	$x\log(x) - x$

Make your own table!




Newton ("Sir Slightly Annoyed") and Leibniz ("Mr Sour Face")

Isaac Newton and **Gottfried Leibniz** are the main figures for the fundamental theorem of calculus. You can see from the expression of their faces, they are not pleased that Oliver has added other calculus pioneers. The sour faces might also have to do with the fact that they have to live already for 10 years on the same handout with **Austin Powers** and **Doctor Evil**. Since 2021, they got reanimated with the Deep Heritage tool, which is an internet sensation these days.



Homework

Problem 18.1: Find a function F such that $F' = f$, then integrate
a) $\int_{-} 1^1 4x^{33} + 300x^2 \, dx$.
b) $\int_0^1 (x+1)^5 \, dx$.

Problem 18.2: Find a function F such that $F' = f$, then integrate:
a) $\int_2^3 5/(x-1) \, dx$, b) $\int_0^{\sqrt{\pi}} \sin(x^2) 4x \, dx$

Problem 18.3: Evaluate the following integrals:
a) $\int_1^2 2^x \, dx$
b) $\int_0^{\sqrt{3}} \frac{1}{1+x^2} \, dx$,

Problem 18.4: a) Compute $F(x) = \int_0^{x^3} \sin(t) \, dt$, then find $F'(x)$.
b) Compute $G(x) = \int_{\sin(x)}^{\cos(x)} \exp(t) \, dt$ then find $G'(x)$

Problem 18.5: a) **A clever integral:** Evaluate the following integral (just by being clever, there is no algebra, and no work is needed):

$$\int_{-\pi}^{\pi} \sin(\sin(\sin(\sin(\sin(x))))) \, dx \, .$$

b) **An evil integral**: Evaluate $\int_e^{e^e} \frac{1}{\log(x)x} \, dx$.
Hint: Figure out a function $F(x)$ which satisfies $F'(x) = 1/(\log(x)x)$. Don't hesitate to ask MiniMe (Oliver).

Clever and evil

INTRODUCTION TO CALCULUS

MATH 1A

Unit 19: Anti-derivatives

Lecture

19.1. The definite integral $\int_a^b f(t)\ dt$ represents a **signed area under the curve**. We say "signed" because the area of the region below the curve is counted negatively. There is something else to mention:

> **Definition:** For every C, the function $F(x) = \int_0^x f(t)\ dt + C$ is called an **anti-derivative** of g. The constant C is arbitrary and not fixed. When

19.2. The fundamental theorem of calculus assured us that

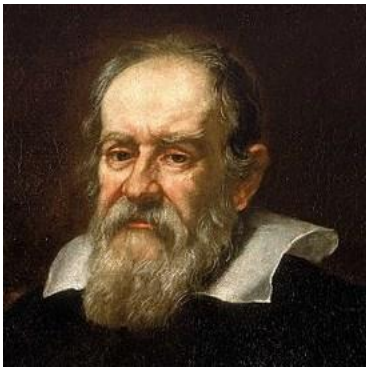
> The anti derivative gives us from a function f a function F which has the property that $F' = f$. Two different anti derivatives F differ only by a constant.

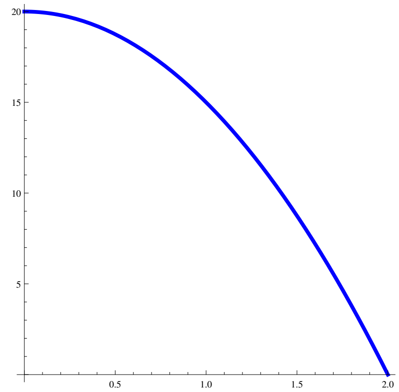
19.3. Finding the anti-derivative of a function is in general harder than finding the derivative. We will learn some techniques but it is in general not possible to give anti derivatives for a function, if it looks simple.

Example: Find the anti-derivative of $f(x) = \sin(4x) + 20x^3 + 1/x$. Solution: We can take the anti-derivative of each term separately. It is $F(x) = -\cos(4x)/4 + 4x^4 + \log(x) + C$.

Example: Find the anti-derivative of $f(x) = 1/\cos^2(x) + 1/(1-x)$. **Solution:** we can find the anti-derivatives of each term separately and add them up. The result is $F(x) = \tan(x) + \log|1-x| + C$.

Example: **Galileo** measured **free fall**, a motion with constant acceleration. Assume $s(t)$ is the height of the ball at time t. Assume the ball has zero velocity initially and is located at height $s(0) = 20$. We know that the velocity is $v(t)$ is the derivative of $s(t)$ and the acceleration $a(t)$ is constant equal to -10. So, $v(t) = -10t + C$ is the antiderivative of a. By looking at v at time $t = 0$ we see that $C = v(0)$ is the initial velocity and so zero. We know now $v(t) = -10t$. We need now to compute the anti derivative of $v(t)$. This is $s(t) = -10t^2/2 + C$. Comparing $t = 0$ shows $C = 20$. Now $s(t) = 20 - 5t^2$. The graph of s is a parabola. If we give the ball an additional horizontal velocity, such that time t is equal to x then $s(x) = 20 - 5x^2$ is the visible trajectory. We see that jumping from 20 meters leads to a fall which lasts 2 seconds.





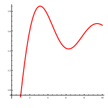
Example: The **total cost** is the anti-derivative of the **marginal cost** of a good. Both the marginal cost as well as the total cost are a function of the quantity produced. For instance, suppose the total cost of making x shoes is 300 and the total cost of making $x + 4$ shoes is 360 for all x. The marginal cost is $60/4 = 15$ dollars. In general the marginal cost changes with the number of goods. There is additional cost needed to produce one more shoe if 300 shoes are produced. **Problem:** Assume the marginal cost of a book is $f(x) = 5 - x/100$ and that producing the first 10 books costs 1000 dollars. What is the total cost of producing 100 books? **Answer:** The anti derivative $5 - x/100$ of f is $F(x) = 5x - x^2/100 + C$ where C is a constant. By comparing $F(10) = 1000$ we get $50 - 100/100 + C = 1000$ and so $C = 951$. the result is $951 + 5 * 100 - 10'000/100 = 1351$. The average book prize has gone down from 100 to 13.51 dollars.
Example: The **total revenue** $F(x)$ is the anti-derivative of the **marginal revenue** $f(x)$. Also these functions depend on the quantity x produced. We have $F(x) = P(x)x$, where $P(x)$ is the prize. Then $f(x) = F'(x) = P'(x)x + P$. For a **perfect competitive market**, $P'(x) = 0$ so that the prize is equal to the marginal revenue.

Definition: A function f is called **elementary**, if it can be constructed using addition, subtraction, multiplication, division, compositions from polynomials or roots. In other words, an elementary function is built up with functions like x^3, $\sqrt{\cdot}$, $\exp, \log, \sin, \cos, \tan$ and $\arcsin, \arccos, \arctan$.

Example: The function $f(x) = \sin(\sin(\pi + \sqrt{x} + x^2)) + \log(1 + \exp((x^6 + 1)/(x^2 + 1)) + (\arctan(e^x))^{1/3}$ is an elementary function.
Example: The anti derivative of the sinc function is called the **sine-integral**

$$Si(x) = \int_0^x \frac{\sin(t)}{t} \, dt \; .$$

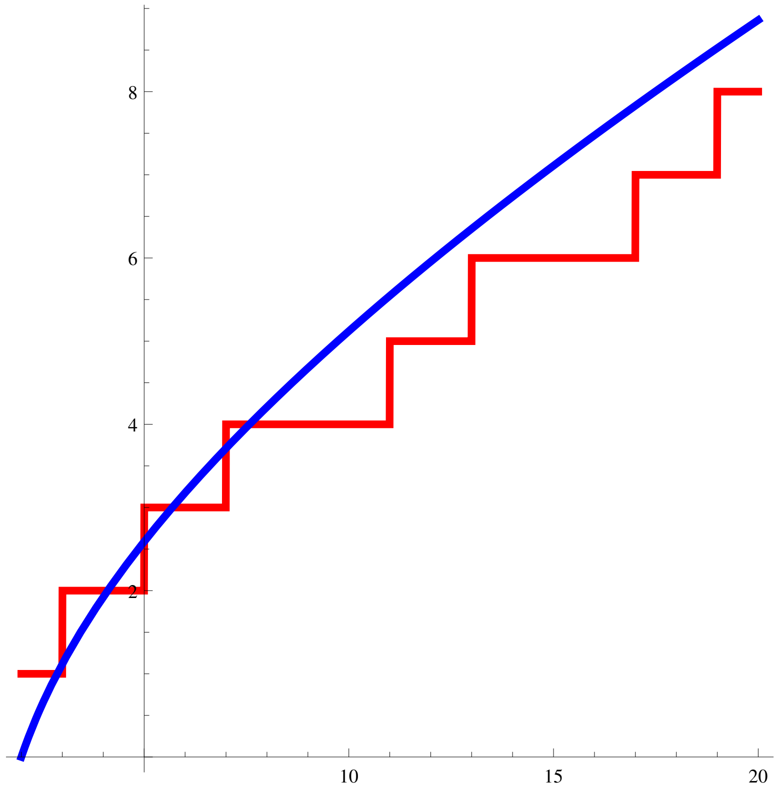
The function $Si(x)$ is not an elementary function.



Example: The **offset logarithmic integral** is defined as

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log(t)}$$

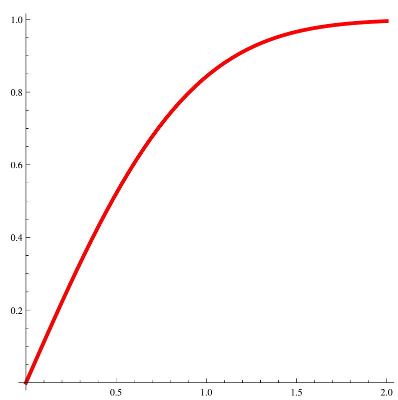
It is a specific anti-derivative. It is a good approximation of the number of prime numbers less than x. The graph below illustrates this. The second stair graph shows the number $\pi(x)$ of primes below x. For example, $\pi(10) = 4$ because $2, 3, 5, 7$ are the only primes below it. The function Li is not an elementary function.



Example: The **error function**

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

is important in statistics. It is not an elementary function.



The Mathematica command "Integrate" uses about 500 pages of Mathematica code and 600 pages of C code. [1] Before software was doing this, tables of integrals were used. These were thousands of pages thick books contains some integrals, which computer

[1]http://reference.wolfram.com/legacy/v3/MainBook/A.9.5.html

algebra systems have trouble with.

There are other integrals we can do, but Mathematica can not do. Especially definite integrals like $\int_0^{2\pi} \sin(\sin(\sin(x)))\, dx$.

Numerical evaluation

What do we do when we have can not find the integral analytically? We can still compute it numerically. Here is an example: the function $f(x) = \sin(\sin(x))$ also does not have an elementary anti-derivative. But you could compute the integral $\int_0^x f(x)\, dx$ numerically with a computer algebra system like Mathematica:

```
NIntegrate[ Sin[Sin[x]],{x,0,10}]
```

One can approximate such a function also using trigonometric Polynomials and then integrate those. In the case, $\sin(\sin(x))$, the function $0.88Sin[x]+0.04Sin[3x]$ is already very close.

Pillow problems

We have wellness day on Tuesday and therefore do not assign homework. Have a day of rest. But here are some integration riddles. We will learn techniques to deal with them. They make also good pillow problems, problems to think about while falling asleep. Try it. Sometimes, you might know the answer in the morning. Maybe you can guess a function which has $f(x)$ as a derivative.

Problem 19.1: $f(x) = \cos(\log(x))/x$.

Problem 19.2: $f(x) = \frac{1}{x^4-1}$.

Problem 19.3: $f(x) = \cot^2(x)$.

Problem 19.4: $f(x) = \cos^4(x)$.

Problem 19.5: $f(x) = \frac{1}{x\log(x)\log(\log(x))}$.

Oliver Knill, knill@math.harvard.edu, Math 1a, Harvard College, Spring 2021

INTRODUCTION TO CALCULUS

MATH 1A

Unit 20: Area

Lecture

20.1. If $f(x) \geq 0$, then $\int_a^b f(x)\ dx$ is the **area under the graph** of $f(x)$ and above the interval $[a, b]$ on the x-axis. If the function is negative, then $\int_a^b f(x)\ dx$ is negative too and the integral is minus the area below the curve:

Therefore, $\int_a^b f(x)\ dx$ is the difference of the area above the graph minus the area below the graph. We call it a **signed area**.

20.2. More generally we can also look at **areas sandwiched between two graphs** f **and** g.

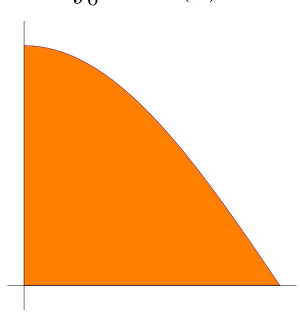
The area of a region G enclosed by two graphs $f \leq g$ and bound by $a \leq x \leq b$ is

$$\int_a^b g(x) - f(x)\ dx$$

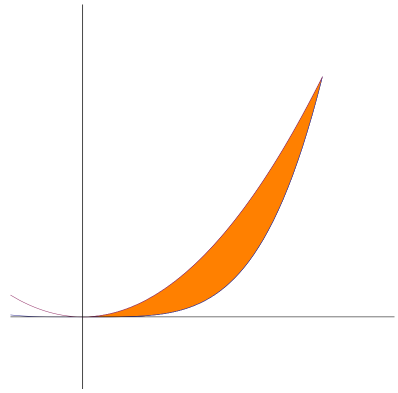
.

20.3. Make sure that if you have to compute such an integral that $g \geq f$ before giving it the interpretation of an area.

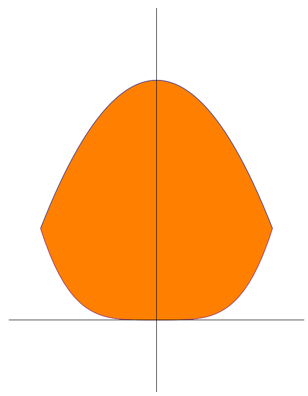
Example: Find the area of the region bound by the cos function and the x and y axes. cos function. **Solution**: $\int_0^{\pi/2} \cos(x)\ dx = 1$.



Example: Find the area of the region enclosed by the graphs $f(x) = x^2$ and $f(x) = x^4$.



Example: Find the area of the region enclosed by the graphs $f(x) = 1 - x^2$ and $g(x) = x^4$. **Solution**: The intersection points are $\pm(\sqrt{5}-1)/2$ and called golden ratio. Now it is routine.

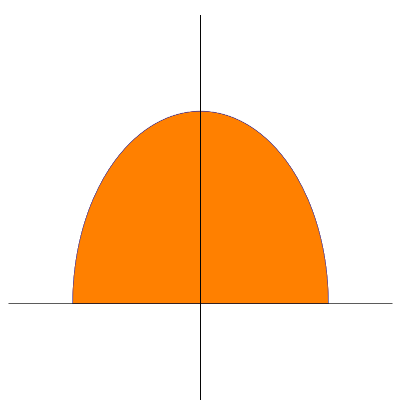


Example: Find the area of the region enclosed by a half circle of radius 1. **Solution**: The half circle is the graph of the function $f(x) = \sqrt{1-x^2}$. The area under the graph is

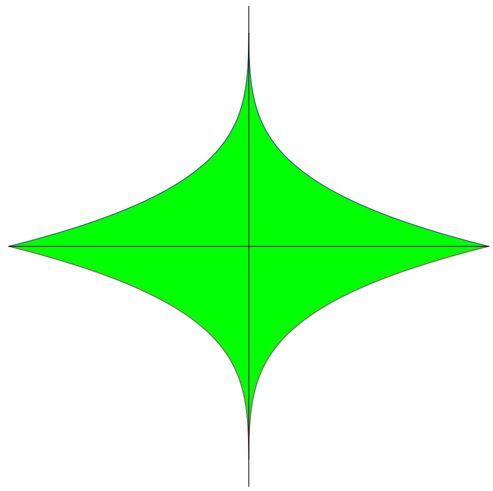
$$\int_{-1}^{1} \sqrt{1-x^2}\, dx \ .$$

Finding the anti-derivative is not so easy. We will find techniques to do so, for now we just are told to look at the derivative of $x\sqrt{1-x^2} + \arcsin(x)$ and see what happens. With this "inspiration", we find the anti derivative to be $(x\sqrt{1-x^2} + \arcsin(x))/2$. The area is therefore

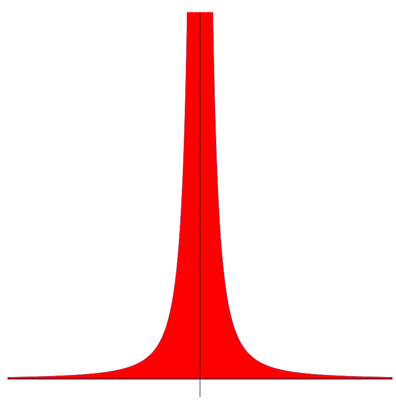
$$\frac{x\sqrt{1-x^2} + \arcsin(x)}{2}|_{-1}^{1} = \frac{\pi}{2} \ .$$



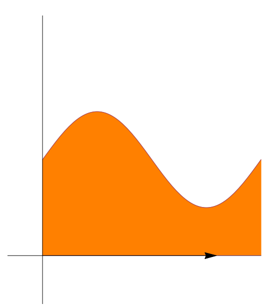
Example: Find the area of the region between the graphs of $f(x) = 1 - |x|^{1/4}$ and $g(x) = -1 + |x|^{1/4}$.



Example: Find the area under the curve of $f(x) = 1/x^2$ between -6 and 6. Naive solution attempt. $\int_{-6}^{6} x^{-2}\, dx = -x^{-1}|_{-6}^{6} = -1/6 - 1/6 = -1/3$. There is something fishy with this computation because $f(x)$ is non-negative so that the area should be positive. But we obtained a negative answer. What is going on?



Example: Find the area between the curves $x = 0$ and $x = 2 + \sin(y)$, $y = 2\pi$ and $y = 0$. **Solution:** We turn the picture by 90 degrees so that we compute the area under the curve $y = 0, y = 2 + \sin(x)$ and $x = 2\pi$ and $x = 0$.



Example: The grass problem. Find the area between the curves $|x|^{1/3}$ and $|x|^{1/2}$.
Solution. This example illustrates how important it is to have a picture. This is good advise for any "word problem" in mathematics.

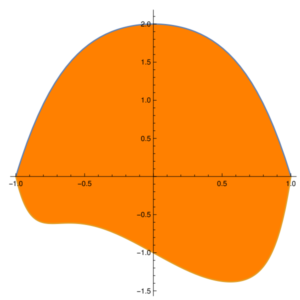
Use a picture of the situation while doing the computation.

Homework

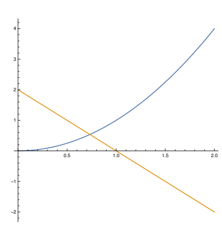
Problem 20.1: Find the area of the bounded region enclosed by the graphs $f(x) = x^4 - 6$ and $g(x) = x^2 + 6$ for $x > 0$. It is a good idea to make a picture.

Problem 20.2: Find the area of the region enclosed by the curves $x = 0, x = \pi/2, y = 4 + \sin(11x), y = \sin^2(2x)$.

Problem 20.3: Find the area of the region enclosed by the graphs $2 - x^4 - x^2$ and $x^{10} - 1 + x^3 - x$.



Problem 20.4: Find the area of the region enclosed by the three curves $y = x^2, y = 2 - 2x$ and $y = 0$.



Problem 20.5: Write down an integral which gives the area of the **area 51** region $x^2 + |y|^{51} \leq 1$ by writing the region as a sandwich between two graphs. Evaluate the integral numerically using Wolfram alpha, Mathematica or any other software.

Oliver Knill, knill@math.harvard.edu, Math 1a, Harvard College, Spring 2021

INTRODUCTION TO CALCULUS

MATH 1A

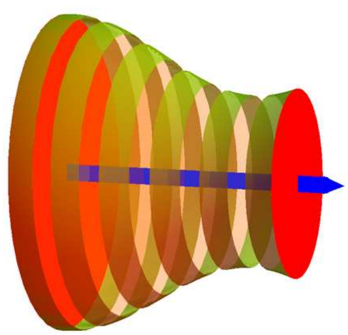
Unit 21: Volume

LECTURE

21.1. To compute the **volume of a solid**, one can cut it into slices, so that each slice is perpendicular to a given line x. If $A(x)$ is the **area of the slice** and the body is enclosed between a and b then

$$V = \int_a^b A(x)\, dx$$

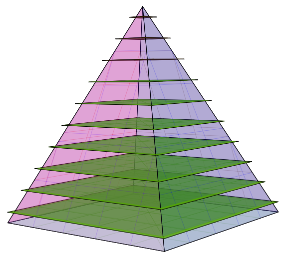
is the **volume** of the body. The integral adds up $A(x)dx$, the volume of the slices.



Example: Compute the volume of a pyramid with square base length 2 and height 2. **Solution**: we can assume the pyramid is built over the square $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. The cross section area at height h is $A(h) = (2-h)^2$. Therefore,

$$V = \int_0^2 (2-h)^2\, dh = \frac{8}{3}\ .$$

This is base area 4 times height 2 divided by 3.



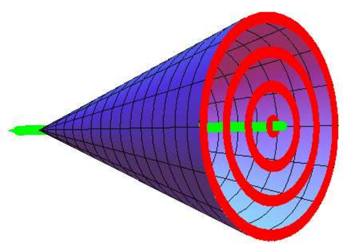
Definition: A **solid of revolution** is a surface obtained by rotating the graph of a function $f(x)$ around the x-axis.

The area of the cross section at x of a solid of revolution is $A(x) = \pi f(x)^2$. The volume of the solid is $\int_a^b \pi f(x)^2\, dx$.

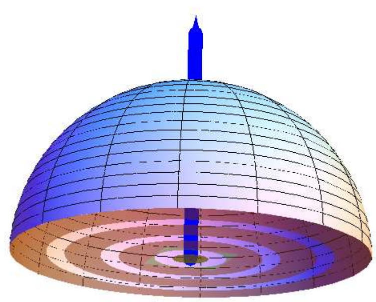
Example: Find the volume of a **round cone** of height 2 and where the circular base has the radius 1. **Solution**. This is a solid of revolution obtained by rotation the graph of $f(x) = x/2$ around the x axes. The area of a cross section is $\pi x^2/4$. Integrating this up from 0 to 2 gives

$$\int_0^2 \pi x^2/4\, dx = \frac{x^3}{4 \cdot 3}|_0^2 = \frac{2\pi}{3}\ .$$

This is the height 2 times the base area π divided by 3.

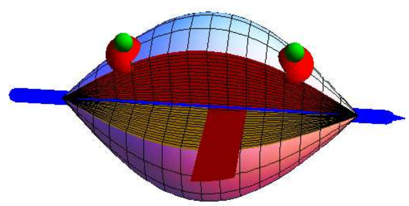


Example: Find the volume of a **half sphere** of radius 1. **Solution:** The area of the cross section at height h is $\pi(1 - h^2)$.



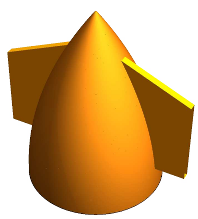
Example: If the function $f(x) = \sin(x)$ is rotated around the x axes, we get a **lemon**. But now we cut out a slice of $60 = \pi/3$ degrees as in the picture. Find the volume of the solid.

Solution: The area of a slice without the missing piece is $\pi \sin^2(x)$. The integral $\int_0^\pi \sin^2(x)\ dx$ is $\pi/2$ as derived in the lecture. Having cut out 1/6'th the area is $(5/6)\pi \sin^2(x)$. The volume is $\int_0^\pi (5/6)\pi \sin(x)^2\ dx = (5/6)\pi^2/2$.



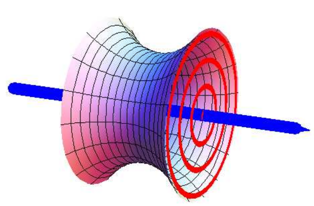
Homework

Problem 1: Space Ship SN10 just passed a high altitude test. We model the top of the rocket with a solid. Compute its volume for which the radius at position x is $9 - x^2$ and x ranges from 0 to 3.



Problem 2: A **catenoid** is the surface obtained by rotating the graph of $f(x) = \cosh(x) = (\exp(x) + \exp(-x))/2$ around the x-axes. We have seen that the graph of f is the chain curve, the shape of a hanging chain. Find the volume of of the solid enclosed by the catenoid between $x = -3$ and $x = 3$.

Hint. You might want to check first the identity $\cosh(x)^2 = (1 + \cosh(2x))/2$ using the definition $\cosh(x) = (\exp(x) + \exp(-x))/2$.



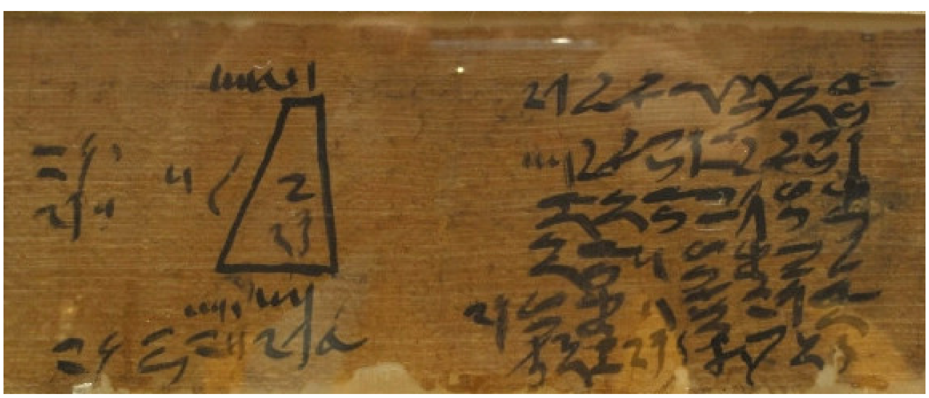
Problem 3: A **tomato** is given by $z^2 + x^2 + 4y^2 = 1$. If we slice perpendicular to the y axes, we get a circular slice $z^2 + x^2 \leq 1 - 4y^2$ of radius $\sqrt{1-4y^2}$. Find the area of this slice, then determine the volume of the tomato.

Problem 4: **Archimedes** was so proud of his formula for the volume of a sphere that he wanted the formula displayed on his tomb stone. To derive the formula, he wrote the volume of a half sphere of radius 1 as the difference between the volume of a cylinder of radius 1 and height 1 and the volume of a cone of base radius 1 and height 1. Relate the cross section area of the cylinder-cone complement with the cross section area of the sphere to recover his argument! No credit is given for screaming "Eureka".

Problem 5: Volumes were among the first quantities, Mathematicians wanted to measure and compute. One problem on **Moscow Egypt papyrus** dating back to 1850 BC explains the general formula $h(a^2 + ab + b^2)/3$ for a **truncated pyramid** with base length a, roof length b and height h. Verify that if you slice such a **frustrum** at height x, the area is $A(x) = (a + (b-a)x/h)^2$. Now use this to compute the volume using calculus.

Here is the translated formulation from the papyrus: [1] [2]

Remark: "You are given a truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top. You are to square this 4 result 16. You are to double 4 result 8. You are to square 2, result 4. You are to add the 16, the 8 and the 4, result 28. You are to take one-third of 6 result 2. You are to take 28 twice, result 56. See it is 56. You will find it right".



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[1] H. Eves, Great Moments in Mathematics, Vol. 1, MAA, Dolciani Math. Expos., 1980, p. 10

[2] Image Source: Carles Dorce, https://themathematicaltourist.wordpress.com

Unit 22: Improper Integrals

Lecture

22.1. Integrals on infinite intervals or integrals with a function becoming infinite at some point are called **improper integrals**. The area under the curve can either remain finite or become infinite. Here is an example, where the value is finite:

Example: What is the integral

$$\int_1^\infty \frac{1}{x^4}\, dx\ ?$$

Since the anti-derivative is $-1/(3x^3)$, we have

$$\frac{-1}{3x^3}|_1^\infty = -1/(3\infty) + (1/3) = 1/3\ .$$

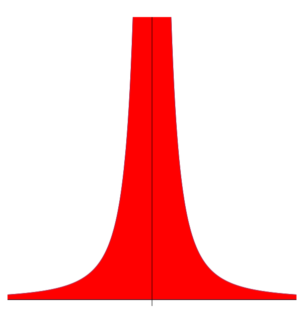
To justify this, compute the integral $\int_1^b 1/x^4 dx = (1/3 - 1/(3b^3))$ and see that in the limit $b \to \infty$, the value $1/3$ is achieved.

22.2. A shocking example is the following. It is a case where things go wrong.

Example:

$$\int_{-1}^1 \frac{1}{x^2}\, dx = -\frac{1}{x}|_{-1}^1 = -1 - 1 = -2\ .$$

This does not make any sense because the function is positive so that the integral should be a positive area. The problem is this time not at the boundary $-1, 1$. The sore point is $x = 0$ over which we have carelessly integrated over.



22.3. The next example illustrates the problem with the previous example better:
Example: The computation

$$\int_0^1 \frac{1}{x^2}\, dx = -\frac{1}{x}|_0^1 = -1 + \infty \ .$$

indicates that the integral does not exist. We can justify by looking at integrals

$$\int_a^1 \frac{1}{x^2}\, dx = -\frac{1}{x}|_a^1 = -1 + \frac{1}{a}$$

which are fine for every $a > 0$. But this does not converge for $a \to 0$. Now

$$\int_{-1}^{-a} \frac{1}{x^2}\, dx = -\frac{1}{x}|_{-1}^{-a} = -1 + \frac{1}{a} \ .$$

If we add up both, we get $-2 + 2/a$. This value is positive for every $0 < a < 1$ but it does not disappear for $a \to 0$.

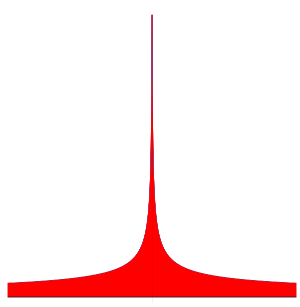
22.4. Do we always have a problem if the function goes to infinity at some point?
Example: Find the following integral

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx \ .$$

Solution: Since the point $x = 0$ is problematic, we integrate from a to 1 with positive a and then take the limit $a \to 0$. Since $x^{-1/2}$ has the anti-derivative $x^{1/2}/(1/2) = 2\sqrt{x}$, we have

$$\int_a^1 \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x}|_a^1 = 2\sqrt{1} - 2\sqrt{a} = 2(1 - \sqrt{a}) \ .$$

There is no problem with taking the limit $a \to 0$. The answer is 2. Even so the region is infinite its area is finite. This is an interesting example. Imaging this to be a container for paint. We can fill the container with a finite amount of paint but the wall of the region has infinite length.



Example: Evaluate the integral $\int_0^1 1/\sqrt{1-x^2}\, dx$. **Solution:** The anti-derivative is $\arcsin(x)$. In this case, it is not the point $x = 0$ which produces the difficulty. It is the point $x = 1$. Take $a > 0$ and evaluate

$$\int_0^{1-a} \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin(x)|_0^{1-a} = \arcsin(1-a) - \arcsin(0) \ .$$

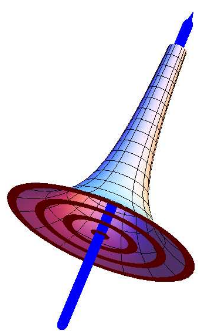
Now $\arcsin(1-a)$ has no problem at limit $a \to 0$. Since $\arcsin(1) = \pi/2$ exists. We get therefore the answer $\arcsin(1) = \pi/2$.

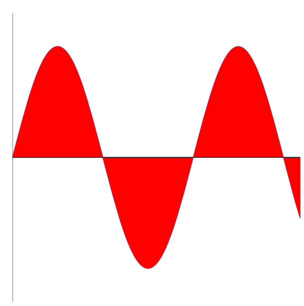
Example: Rotate the graph of $f(x) = 1/x$ around the x-axes and compute the volume of the solid between 1 and ∞. The cross section area is π/x^2. If we look at the integral from 1 to a fixed R, we get

$$\int_1^R \frac{\pi}{x^2} \, dx = -\frac{\pi}{x}|_1^R = -\pi/R + \pi \; .$$

This converges for $R \to \infty$. The volume is π. This famous solid is called **Gabriel's trumpet**. This solid is so prominent because if you look at the surface area of the small slice, then it is larger than $dx 2\pi/x$. The total surface area of the trumpet from 1 to R is therefore larger than $\int_1^R 2\pi/x \, dx = 2\pi(\log(R) - \log(1))$. which goes to infinity. We can **fill** the trumpet with a finite amount of paint but we can not **paint** its surface.

Example: Evaluate the integral $\int_0^\infty \sin(x) \, dx$. **Solution.** There is no problem at the boundary 0 nor at any other point. We have to investigate however, what happens at ∞. Therefore, we look at the integral $\int_0^b \sin(x) \, dx = -\cos(x)|_0^b = 1 - \cos(b)$. We see that the limit $b \to \infty$ does not exist. The integral fluctuates between 0 and 2.





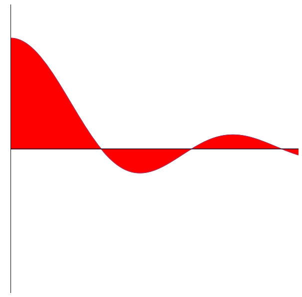
22.5. The next example leads to a topic in a follow-up course. It is not covered here, but could make you curious:

Example: What about the integral

$$I = \int_0^\infty \frac{\sin(x)}{x} \, dx \; ?$$

Solution. The anti derivative is the Sine integral $Si(x)$ so that we can write $\int_0^b \sin(x)/x \, dx = Si(b)$. It turns out that the limit $b \to \infty$ exists and is equal to $\pi/2$ but this is a topic for a second semester course like Math 1b. The integral can be written as an alternating series, which converges and there are many ways to compute it: [1]

[1] Hardy, Mathematical Gazette, 5, 98-103, 1909.



22.6.

$\int_a^\infty f(x)\,dx$ id defined as $\lim_{t\to\infty}\int_a^t f(x)\,dx$ if the limit exists.
$\int_0^b f(x)\,dx$ is defined as $\lim_{t\to 0}\int_t^b f(x)\,dx$ if the limit exists

Homework

Problem 22.1: Evaluate the integral $\int_0^1 x^{-7/11}\,dx$.

Problem 22.2: For which $0 < p < \infty$ does the integral $\int_1^\infty 1/x^p\,dx$ exist? To investigate this, look at $\int_1^t 1/x^p\,dx$ and decide for which p, the limit $t\to\infty$ exists. If you are unsure what do do, look at cases like $p = 1/3, p = 1/2, p = 1, p = 3/2, p = 2, p = 3$.

Problem 22.3: Evaluate the improper integral $\int_{-1}^1 \frac{1}{\pi\sqrt{1-x^2}}\,dx$.
This example is related to the arcsin distribution in probability theory. Guess where this name come from?

Problem 22.4: Evaluate the integral $\int_{-3}^4 (x^2)^{1/3}\,dx$. To make sure that the integral is fine, check separately whether $\int_{-3}^0$ and $\int_0^4$ work.

The integral $\int_{-2}^1 1/x\,dx$ does not exist. We can however take a positive $a > 0$ and look at

$$\int_{-2}^{-a} 1/x\,dx + \int_a^1 1/x\,dx = \log|a| - \log|-2|) + (\log|1| - \log|a|) = \log(2)\ .$$

If the limit exists, it is called the **Cauchy principal value** of the improper integral.

Problem 22.5: Find the Cauchy principal value of

$$\int_{-3}^4 1/x^3\,dx\ .$$

Unit 23: PDF and CDF

Lecture

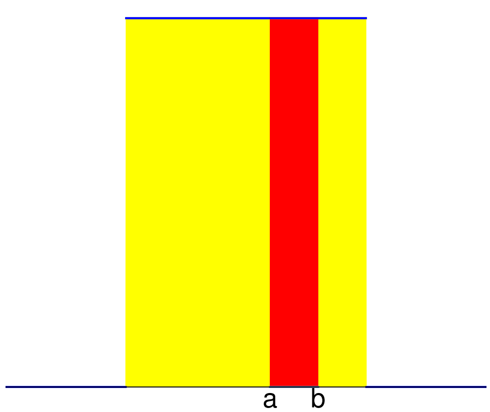
23.1. In **probability theory** one considers functions too:

Definition: A non-negative piece-wise continuous function $f(x)$ which has the property that $\int_{-\infty}^{\infty} f(x)\, dx = 1$ is called a **probability density function**. [1]. For every interval $A = [a, b]$, the number

$$\mathrm{P}[A] = \int_a^b f(x)\, dx$$

is the **probability** of the event that the data are in A.

23.2. An important case is the function $f(x)$ which is 1 on the interval $[0, 1]$ and 0 else. It is the **uniform distribution** on $[0, 1]$. **Random number generators** in computers first of all generate random numbers with that distribution. In Mathematica, you get such numbers by evaluating **Random**[]. In Python you get it with **import random; random.uniform(0,1)**. The probability $\int_{0.3}^{0.7} f(x)\, dx$ for example is 0.4. Here is the **function** $f(x)$:

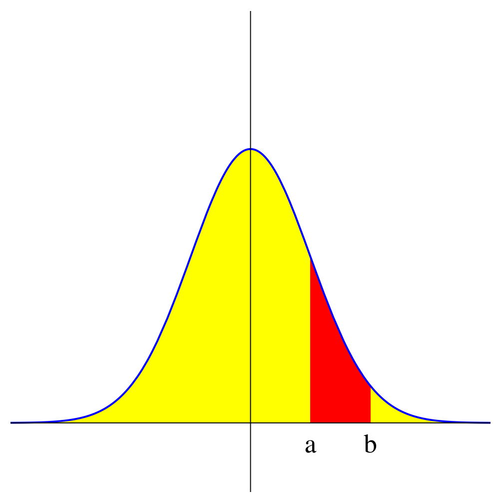


23.3. An other important probability density is the **standard normal distribution**, also called **Gaussian distribution**. You can get such random numbers in python by **import random; random.gauss(0,1)**.

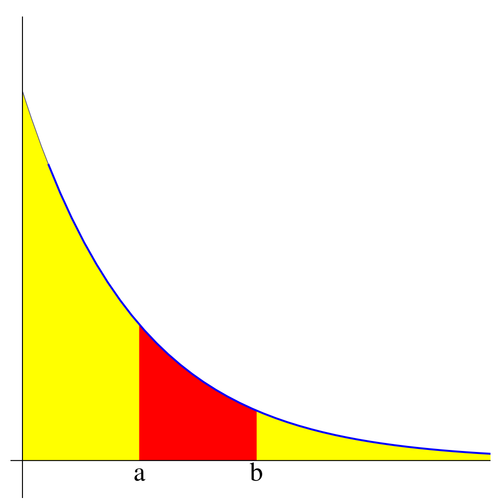
Definition: The **normal distribution** has the density

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \; .$$

23.4. It is the distribution which appears most often if data can take both positive and negative values. One reason why it appears so often is that if one observes different unrelated quantities then their sum, suitably normalized is close to the normal distribution. **Errors** for example often have normal distribution. Astronomers like Galileo noticed this already in 1630ies. Laplace in 1774 first defined probability distributions and Gauss in 1801 first looked at the normal distribution, also in the context of analyzing astronomical data when searching for the **dwarf planet** Ceres.



Example: The probability density function of the **exponential distribution** is defined as $f(x) = e^{-x}$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$. It is used to used measure lengths of arrival times like the time until you get the next email. The density is zero for negative x because there is no way we can travel back in time.
What is the probability that you get an email between times $x = 1$ and times $x = 2$? Answer: it is $\int_1^2 f(x) \, dx = e^{-1} - e^{-2} = 1/e - 1/e^2$.



Definition: Assume f is a probability density function (PDF). The anti-derivative $F(x) = \int_{-\infty}^{x} f(t)\ dt$ is called the **cumulative distribution function** (CDF).

Example: For the exponential function the cumulative distribution function is

$$\int_{-\infty}^{x} f(x)\ dx = \int_{0}^{x} f(x)\ dx = -e^{-x}|_0^x = 1 - e^{-x}\ .$$

Definition: The probability density function $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$ is called the **Cauchy distribution**.

Example: Find the cumulative distribution function of the Cauchy distribution.
Solution:

$$F(x) = \int_{-\infty}^{x} f(t)\ dt = \frac{1}{\pi}\arctan(x)|_{-\infty}^{x} = (\frac{1}{\pi}\arctan(x) + \frac{1}{2})\ .$$

Definition: The **mean** of a distribution is the number

$$m = \int_{-\infty}^{\infty} xf(x)\ dx\ .$$

Example: The mean of the distribution $f(x_{)} = e^{-x}$ on $[0,\infty)$ is

$$\int_0^{\infty} xe^{-x}\ dx\ .$$

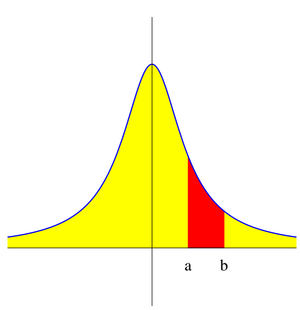
We do not know yet how to compute this but learn a technique later. For now, we have to guess the anti derivative or being told that it is $(-1-x)e^{-x}$. We can check that the derivative of this function is indeed e^{-x}. So,

$$\int_0^{\infty} xe^{-x}\ dx = \lim_{t\to\infty}(-1-x)e^{-x}|_0^t = \lim_{t\to\infty}(-1-t)e^{-t} + 1 = 1\ .$$

23.5. The distribution looks similar to the Gaussian distribution, but it has more risk. The **variance** of this distribution

$$\int_{\infty}^{\infty} x^2 f(x)\ dx = (1/\pi)\int_{-\infty}^{\infty}\frac{x^2}{1+x^2}\ dx$$

is infinite. The function $\frac{x^2}{1+x^2}$ is asymptotically 1 and has a divergent integral from $-\infty$ to ∞.



Homework

Problem 23.1: Assume the probability density for the time you have to wait for your next shopping delivery you get is $f(x) = 7e^{-7x}$ where x is time in hours.
What is the probability that you get your next text message in the next 5 hours but not before 1 hour?

Problem 23.2: Assume the probability distribution for the waiting time to the next rain is $f(x) = (1/5)e^{-x/5}$, where x has days as unit.
What is the probability to get a warm day between tomorrow and after tomorrow that is between $x = 1$ and $x = 2$?

Problem 23.3: a) Verify that the function $f(x)$ which is defined to be zero outside the interval $[-1, 1]$ and given as $\frac{1}{\pi\sqrt{1-x^2}}$ inside the interval $[-1, 1]$ is a probability distribution.
b) What is the cumulative distribution function $F(x)$?
c) What is the expectation $\int_{-1}^{1} xf(x)\, dx$?

Problem 23.4: Assume some risky experiment leads to discrepancies (errors) which are distributed according to the Cauchy distribution $f(x) = 1/(\pi(1 + x^2))$.
a) Find the probability that the error is in absolute value larger than 1.
b) Find the probability that the error is smaller than $-\sqrt{3}/2$. In other words, what is $\int_{-\infty}^{-\sqrt{3}/2} f(x)\, dx$?

Problem 23.5: If $f(x)$ is a probability distribution, then $m = \int_{-\infty}^{\infty} xf(x)\, dx$ is called the **mean** of the distribution. a) Compute the mean for the standard normal distribution.
b) Compute the mean for the Cauchy distribution $f(x) = \frac{1}{\pi} \frac{1}{1+x^2}$.
c) Compute the mean for the Exponential distribution $f(x) = e^{-x}$ on $[0, \infty)$. You might want to guess an anti-derivative of $xf(x)$. .

INTRODUCTION TO CALCULUS

MATH 1A

Unit 24: Substitution

Lecture

24.1. We know so far how to integrate functions like e^{6x} or $1/(1+x)$. The technique of **substitution** allows us to find more complicated anti-derivatives. If we differentiate the function $\sin(x^2)$ and use the chain rule, we get $\cos(x^2)2x$. The fundamental theorem of calculus tells us therefore that the anti-derivative of $\cos(x^2)2x$ is

$$\int \cos(x^2)2x \; dx = \sin(x^2) + C \; .$$

24.2. How can we see the integral without knowing the result already? Here is a very important case:

If $f(x) = g(u(x))u'(x)$, then the anti-derivative of f is $G(u(x)) + C$, where G is the anti-derivative of g.

Example: Find the anti derivative of

$$f(x) = e^{x^4+x^2}(4x^3 + 2x) \; .$$

Solution: It is $e^{x^4+x^2} + C$.
Example: Find

$$\int \sqrt{x^5+1}x^4 \; dx \; .$$

Solution. Try $(x^5+1)^{3/2}$ and differentiate. This gives $15/2$ of what we have. Therefore $F(x) = (2/15)(x^5+1)^{3/2}$.
Example: Find the anti-derivative of

$$\frac{\log(x)}{x} \; .$$

Solution: We spot that $1/x$ is the derivative of $\log(x)$. The anti-derivative is $\log(x)^2/2 + C$.

24.3. Writing down the function and adjusting a constant is the **"speedy rule"**:

If $\int f(ax+b)\,dx = F(ax+b)/a$ where F is the anti derivative of f.

Example: $\int \sqrt{x+1}\,dx$. **Solution:** $(x+1)^{3/2}(2/3)+C$.
Example: $\int \frac{1}{1+(5x+2)^2}\,dx$. **Solution:** $\arctan(5x+2)(1/5)+C$.

24.4. The method of substitution formalizes this: A) select part of the formula, call it u. B) then write $du = u'dx$. C) replace dx with du/u'. D) If all terms x have disappeared, integrate. E) Back substitute the variable x. If things should not work, go back to A) and try an other u.

$$\int f(\,\mathrm{u(x)}\,)\,\mathrm{u'(x)}\quad \mathrm{dx} = \int g(\,\mathrm{u}\,)\,\mathrm{du}\ .$$

24.5. We aim to end up with an integral $\int g(u)\,du$ which does not involve x anymore. Finally, after integration of this integral, do a **back-substitution**: replace the variable u again with the function $u(x)$.

Example: Find the anti-derivative of $\int \log(x)/x\,dx$. **Solution:** Pick $u = log(x), du = (1/x)dx$. Because $dx = xdu$, we get $\int udu = u^2/2+C$. Back substitute to get $\log^2(x)/2+C$.

Example: Find the anti-derivative

$$\int \log(\log(x))\frac{1}{\log(x)x}\,dx\ .$$

Solution: Try $u = \log(x)$ and $du = (1/x)dx$, then plug this into the formula. It gives $\int \log(u)/u\,du$. We have just solved this integral before and got $\log(u)^2/2+C$.

Example: Solve the integral

$$\int \frac{x}{1+x^4}\,dx\ .$$

Solution: Substitute $u = x^2, du = 2xdx$ gives $(1/2)\int du/(1+u^2)\,du = (1/2)\arctan(u) = (1/2)\arctan(x^2)+C$.

Example: What is the anti-derivative of $\sin(\sqrt{x})/\sqrt{x}$?
Solution. Try $u = \sqrt{x}, x = u^2, dx = 2udu$. The result is $-2\cos(\sqrt(x)) + C$.

24.6. Here is an example that is more challenging
Example: Solve the integral

$$\int \frac{x^3}{\sqrt{x^2+1}} \, dx \; .$$

Solution. Trying $u = \sqrt{x^2+1}$ does not work. Try $u = x^2 + 1$, then $du = 2xdx$ and $dx = du/(2\sqrt{u-1})$. Substitute this in to get

$$\int \frac{\sqrt{u-1}^3}{2\sqrt{u-1}\sqrt{u}} \, du = \int \frac{(u-1)}{2\sqrt{u}} = \int u^{1/2}/2 - u^{-1/2}/2 \; du = u^{3/2}/3 - u^{1/2} = \frac{(x^2+1)^{3/2}}{3} - (x^2+1)^{1/2} \; .$$

24.7. When doing **definite integrals** $\int_a^b f(x) \, dx$, we could find the anti-derivative as described and then fill in the boundary points. Substituting the boundaries directly accelerates the process since we do not have to substitute back to the original variables:

$$\int_a^b g(u(x))u'(x) \, dx = \int_{u(a)}^{u(b)} g(u) \, du \; .$$

Proof. This identity follows from the fact that the right hand side is $G(u(b)) - G(u(a))$ by the fundamental theorem of calculus. The integrand on the left has the anti derivative $G(u(x))$. Again by the fundamental theorem of calculus the integral leads to $G(u(b)) - G(u(a))$.

Example: Find the anti-derivative of $\int_0^2 \sin(x^3 - 1)x^2 \, dx$. **Solution**:

$$\int_{x=0}^{x=2} \sin(x^3+1)x^2 \, dx \; .$$

Solution: Use $u = x^3 + 1$ and get $du = 3x^2dx$. We get

$$\int_{u=1}^{u=9} \sin(u)du/3 = (1/3)\cos(u)|_1^9 = [-\cos(9) + \cos(1)]/3 \; .$$

Example: $\int_0^1 \frac{1}{5x+1} \, dx = [\log(u)]/5|_1^6 = \log(6)/5$.
Example: $\int_3^5 \exp(4x - 10) \, dx = [\exp(10) - \exp(2)]/4$.

24.8. Substituting the bounds can sometimes be a bit tricky. An alternative way is to find first the anti derivative and then plug in the original bounds. Avoiding substituting the bounds actually is often the preferred way.
Example: $\int_3^5 \exp(4x - 10) \, dx = F(5) - f(3)$, where $F(x) = \exp(4x - 10)/4$.

Homework

Problem 24.1: Find the following anti-derivatives.
a) $\int x^2 \sin(x^3)\, dx$
b) $\int e^{x^6+x}(6x^5+1)\, dx$
c) $-\cos(\sin(3x))/3$
d) $e^{\tan(2x)}/\cos^2(2x)$.

Problem 24.2: Compute the following definite integrals. It is fine to find first the anti-derivative and only at the end place the bounds:
a) $\int_1^2 \sqrt{x^5+x}(5x^4+1)\, dx$
b) $\int_0^{\sqrt{\pi}} 6\sin(x^2)x\, dx$.
c) $\int_e^{e^2} \frac{\sqrt{\log(x)}}{x}\, dx$.
d) $\int_0^1 \frac{5x}{\sqrt{1+x^2}}\, dx$.

Problem 24.3: Find the definite integral

$$\int_e^{2e} \frac{dx}{\sqrt{\log(x)}x} .$$

Problem 24.4: a) Find the indefinite integral

$$\int \frac{x^5}{\sqrt{x^2+1}}\, dx .$$

b) Find the anti-derivative of

$$f(x) = \frac{1}{x(1+\log(x)^2)} .$$

Problem 24.5: a) Find the anti-derivative of $\cos(x^3)/e^{\sin(x^3)}x^2$.
b) Find the anti-derivative of $\cot(\sqrt{x})/\sqrt{x}$.

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INTRODUCTION TO CALCULUS

MATH 1A

Unit 25: Integration by parts

25.1. Integrating the product rule $(uv)' = u'v + uv'$ gives the method **integration by parts**. It complements the method of substitution we have seen last time. As a rule of thumb, always try first to **1) simplify a function and integrate using known functions**, then **2) try substitution** and finally **3) try integration by parts**.

$$\int \text{u(x)}\ \text{v'}\,(x)dx = u(x)v(x) - \int u'(x)v(x)\, dx.$$

Example: To see how integration by parts work, lets try to find $\int x\sin(x)\, dx$. First identify what you want to differentiate and call it u, the part to integrate is called v'. Now, write down uv and subtract a new integral which integrates $u'v$:

$$\int \text{x}\ \sin(x)\ dx = \text{x}\ (-\cos(x)) - \int 1\ (-\cos(x)\ )\ dx = -x\cos(x) + \sin(x) + C\ dx\ .$$

In class, I will will stream line this by just placing an arrow down under the expression you differentiate and an arrow up under the expression you integrate. You remember to first integrate, then subtract the integral of the expression where you both integrate and differentiate. If you like to write down the u, v, do so and remember $\boxed{\int udv = uv - \int vdu}$.

Example: Find $\int xe^x\, dx$. **Solution.** You want to differentiate x and integrate e^x.

$$\int \text{x}\ \exp(x)\ dx = x\exp(x) - \int 1\cdot\exp(x)\ dx = x\exp(x) - \exp(x) + C\ dx\ .$$

Example: Find $\int \log(x)\, dx$. **Solution.** While there is only one function here, we need two to use the method. Let us look at $\log(x)\cdot 1$:

$$\int \log(x)\ 1\ dx = \log(x)x - \int \frac{1}{x}x\ dx = x\log(x) - x + C\ .$$

Example: Find $\int x\log(x)\, dx$. **Solution.** Since we know from the previous problem how to integrate log we could proceed by taking $x = u$. We can also take $u = \log(x)$ and $dv = x$:

$$\int \log(x)\ \text{x}\ dx = \log(x)\frac{x^2}{2} - \int \frac{1}{x}\frac{x^2}{2}\ dx$$

which is $\log(x)x^2/2 - x^2/4$.

25.2. We see that it is often better to differentiate log first. The word LIATE explained below tells which functions we want to call u and differentiate.

Example: **Marry go round:** Find $I = \int \sin(x)\exp(x)\ dx$. **Solution.** Lets integrate $\exp(x)$ and differentiate $\sin(x)$.

$$= \sin(x)\exp(x) - \int \cos(x)\exp(x)\ dx\ .$$

Lets do it again:

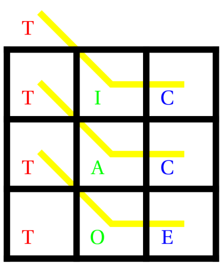
$$= \sin(x)\exp(x) - \cos(x)\exp(x) - \int \sin(x)\exp(x)\ dx.$$

We moved in circles and are stuck! But wait. Are we really? We have now derived an identity

$$I = \sin(x)\exp(x) - \cos(x)\exp(x) - I$$

which we can solve for I and get $\boxed{I = [\sin(x)\exp(x) - \cos(x)\exp(x)]/2.}$

Tic-Tac-Toe



Integration by parts can become complicated if it has to be done several times. Keeping the order of the signs can be especially daunting. Fortunately, there is a powerful **tabular integration by parts method**. It has been called **"Tic-Tac-Toe"** in the movie Stand and deliver. Lets call it **Tic-Tac-Toe** therefore.

Example: Find the anti-derivative of $(x-1)^3 e^{2x}$. **Solution:**

$(x-1)^3$	$\exp(2x)$	
$3(x-1)^2$	$\exp(2x)/2$	$\oplus$
$6(x-1)$	$\exp(2x)/4$	$\ominus$
6	$\exp(2x)/8$	$\oplus$
0	$\exp(2x)/16$	$\ominus$

The anti-derivative is

$$(x-1)^3 e^{2x}/2 - 3(x-1)^2 e^{2x}/4 + 6(x-1)e^{2x}/8 - 6e^{2x}/16 + C\ .$$

Example: Find the anti-derivative of $x^2\cos(x)$. **Solution:**

x^2	$\cos(x)$	
$2x$	$\sin(x)$	$\oplus$
2	$-\cos(x)$	$\ominus$
0	$-\sin(x)$	$\oplus$

The anti-derivative is $x^2 \sin(x) + 2x\cos(x) - 2\sin(x) + C$.

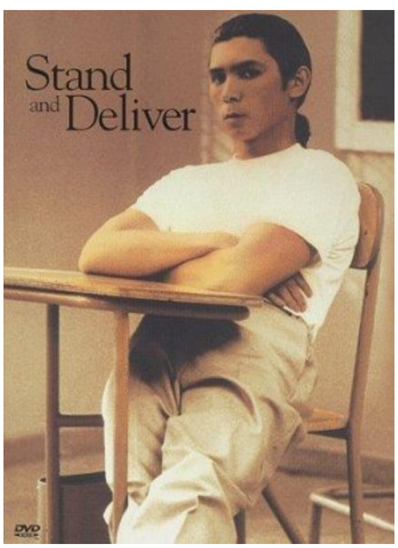
Example: More extreme Find the anti-derivative of $x^7 \cos(x)$. **Solution:**

x^7	$\cos(x)$	
$7x^6$	$\sin(x)$	$\oplus$
$42x^5$	$-\cos(x)$	$\ominus$
$120x^4$	$-\sin(x)$	$\oplus$
$840x^3$	$\cos(x)$	$\ominus$
$2520x^2$	$\sin(x)$	$\oplus$
$5040x$	$-\cos(x)$	$\ominus$
5040	$-\sin(x)$	$\oplus$
0	$\cos(x)$	$\ominus$

The anti-derivative is

$$\begin{aligned} F(x) &= x^7\sin(x) + 7x^6\cos(x) - 42x^5\sin(x) \\ &- 210x^4\cos(x) + 840x^3\sin(x) + 2520x^2\cos(x) \\ &- 5040x\sin(x) - 5040\cos(x) + C \ . \end{aligned}$$

25.3. Do this without this method and you see the value of the method.
[1] [2] [3]:



I myself learned the method from the movie "Stand and Deliver", where **Jaime Escalante** of the Garfield High School in LA uses the method. It can be traced down to an article of V.N. Murty. The method realizes in a clever way an iterated integration by parts method:

$$\begin{aligned} \int fg dx &= fg^{(-1)} - f^{(1)}g^{-2} + f^{(2)}g^{(-3)} - \dots \\ &- (-1)^n \int f^{(n+1)}g^{(-n-1)} \, dx \end{aligned}$$

The method can be verified by induction because the f function is differentiated again and again and the g function is integrated again and again. The alternating minus-plus-signs come from the fact that we subtract again an integral. We always pair a k'th derivative with a $k+1$'th integral and take the sign $(-1)^k$.

Coffee or Tea?

[1] V.N. Murty, Integration by parts, Two-Year College Mathematics Journal 11, 1980, p. 90-94.
[2] D. Horowitz, Tabular Integration by Parts, College Mathematics Journal, 21, 1990, p. 307-311.
[3] K.W. Folley, integration by parts, American Mathematical Monthly 54, 1947, p. 542-543

25.4. When doing integration by parts, We want to try first to differentiate **L**ogs, **I**nverse trig functions, **P**owers, **T**rig functions and **E**xponentials. This can be remembered as **LIPTE** which is close to "lipton" (the tea).

For coffee lovers, there is an equivalent one: **L**ogs, **I**nverse trig functions, **A**lgebraic functions, **T**rig functions and **E**xponentials which can be remembered as **LIATE** which is close to "latte" (the coffee).

Whether you prefer to remember it as a "coffee latte" or a "lipton tea" is up to you.

There is even a better method, the "method of the opportunist":

Just integrate what you can integrate and differentiate the rest.

An don't forget to consider integrating 1, if nothing else works.

Homework

It is the season of holidays worldwide: Passover, Easter, Qingming, Holi, Ramadan, we celebrate wellness day! No homework therefore. We will in Lecture 26 add two or three problems from the following.

Problem 25.1: Integrate $\int x^3 \log(x)\, dx$.

Problem 25.2: Integrate $\int x^5 \sin(x)\, dx$

Problem 25.3: Find the anti derivative of $\int 2x^6 \exp(x)\, dx$.

Problem 25.4: Find the anti derivative of $\int \sqrt{x} \log(x)\, dx$.

Problem 25.5: Find the anti derivative of $\int \sin(x) \exp(-x)\, dx$.

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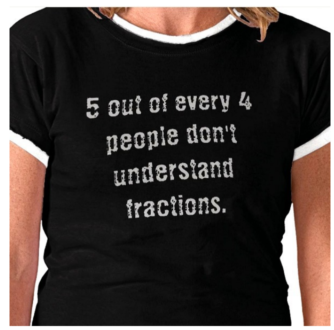
INTRODUCTION TO CALCULUS

MATH 1A

Unit 26: Partial fractions

Lecture



The method of partial fractions is not really about integration. It is about algebra. We have learned how to integrate polynomials like $x^4 + 5x + 3$. What about rational functions? We will see here that they are a piece of cake if you know a bit about algebra.

26.1. Lets see what we know already:

- We also know that integrating $1/x$ gives $\log(x)$. We can for example integrate
$$\int \frac{1}{x-6}\, dx = \log(x-6) + C \ .$$
- We also have learned how to integrate $1/(1+x^2)$. It was an important integral:
$$\int \frac{1}{1+x^2}\, dx = \arctan(x) + C \ .$$
Using substitution, we can do more like
$$\int \frac{dx}{1+4x^2} = \int \frac{du/2}{1+u^2} = \arctan(u)/2 = \arctan(2x)/2 \ .$$
- We also know how to integrate functions of the type $x/(x^2+c)$ using substitution. We can write $u = x^2 + c$ and get $du = 2xdx$ so that
$$\int \frac{x}{x^2+c}\, dx = \int \frac{1}{2u}\, du = \frac{\log(x^2+c)}{2} \ .$$
- Also functions $1/(x+c)^2$ can be integrated using substitution. With $x + c = u$ we get $du = dx$ and
$$\int \frac{1}{(x+c)^2}\, dx = \int \frac{1}{u^2} du = -\frac{1}{u} + C = -\frac{1}{x+c} + C \ .$$

26.2. We would love to be able to integrate any rational function

$$f(x) = \frac{p(x)}{q(x)} ,$$

where p, q are polynomials. This is where **partial fractions come in**. The idea is to write a rational function as a sum of fractions we know how to integrate. The above examples have shown that we can integrate $a/(x+c)$, $(ax+b)/(x^2+c)$, $a/(x+c)^2$ and cases, which after substitution are of this type.

> **Definition:** The **partial fraction method** writes $p(x)/q(x)$ as a sum of functions of the above type which we can integrate.

26.3. This is an algebra problem. Here is an important special case:

> In order to integrate $\int \frac{1}{(x-a)(x-b)} \, dx$, write
>
> $$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} .$$
>
> and solve for A, B.

26.4. In order to solve for A, B, write the right hand side as one fraction again

$$\frac{1}{(x-a)(x-b)} = \frac{A(x-b) + B(x-a)}{(x-a)(x-b)} .$$

We only need to look at the nominator:

$$1 = Ax - Ab + Bx - Ba .$$

In order that this is true we must have $A + B = 0, Ab - Ba = 1$. This allows us to solve for A, B.

Examples

Example: To integrate $\int \frac{2}{1-x^2} \, dx$ we can write

$$\frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$$

and integrate each term

$$\int \frac{2}{1-x^2} = \log(1+x) - \log(1-x) .$$

Example: Integrate $\frac{5-2x}{x^2-5x+6}$. **Solution.** The denominator is factored as $(x-2)(x-3)$. Write

$$\frac{5-2x}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2} .$$

Now multiply out and solve for A, B:

$$A(x-2) + B(x-3) = 5 - 2x .$$

This gives the equations $A + B = -2, -2A - 3B = 5$. From the first equation we get $A = -B - 2$ and from the second equation we get $2B + 4 - 3B = 5$ so that $B = -1$ and so $A = -1$. We have not obtained

$$\frac{5 - 2x}{x^2 - 5x + 6} = -\frac{1}{x - 3} - \frac{1}{x - 2}$$

and can integrate:

$$\int \frac{5 - 2x}{x^2 - 5x + 6}\, dx = -\log(x - 3) - \log(x - 2) \ .$$

Actually, we could have got this one also with substitution. How?

Example: Integrate $f(x) = \int \frac{1}{1-4x^2}\, dx$. **Solution.** The denominator is factored as $(1 - 2x)(1 + 2x)$. Write

$$\frac{A}{1 - 2x} + \frac{B}{1 + 2x} = \frac{1}{1 - 4x^2} \ .$$

We get $A = 1/4$ and $B = -1/4$ and get the integral

$$\int f(x)\, dx = \frac{1}{4}\log(1 - 2x) - \frac{1}{4}\log(1 + 2x) + C \ .$$

26.5. There is a fast method to get the coefficients:

If a is different from b, then the coefficients A, B in

$$\frac{p(x)}{(x - a)(x - b)} = \frac{A}{x - a} + \frac{B}{x - b} \ ,$$

are

$$A = \lim_{x \to a}(x - a)f(x) = p(a)/(a - b),\ B = \lim_{x \to b}(x - b)f(x) = p(b)/(b - a) \ .$$

Proof. If we multiply the identity with $x - a$ we get

$$\frac{p(x)}{(x - b)} = A + \frac{B(x - a)}{x - b} \ .$$

Now we can take the limit $x \to a$ without peril and end up with $A = p(a)/(x - b)$.

26.6. Cool, isn't it? This **Hospital method** or **residue method** saves time especially with many functions where we would a complicated system of linear equations would have to be solved. I highly recommend you use this method.

Math is all about elegance. Avoid complicated methods if simple ones are available.

26.7. Here are examples:

Example: Find the anti-derivative of $f(x) = \frac{2x+3}{(x-4)(x+8)}$. **Solution.** We write

$$\frac{2x+3}{(x-4)(x+8)} = \frac{A}{x-4} + \frac{B}{x+8}$$

Now $A = \frac{2*4+3}{4+8} = 11/12$, and $B = \frac{2*(-8)+3}{(-8-4)} = 13/12$. We have

$$\frac{2x+3}{(x-4)(x+8)} = \frac{(11/12)}{x-4} + \frac{(13/12)}{x+8} .$$

The integral is

$$\frac{11}{12}\log(x-4) + \frac{13}{12}\log(x+8) .$$

Example: Find the anti-derivative of $f(x) = \frac{x^2+x+1}{(x-1)(x-2)(x-3)}$. **Solution.** We write

$$\frac{x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

Now $A = \frac{1^2+1+1}{(1-2)(1-3)} = 3/2$ and $B = \frac{2^2+2+1}{(2-1)(2-3)} = -7$ and $C = \frac{3^2+3+1}{(3-1)(3-2)} = 13/2$. The integral is

$$\frac{3}{2}\log(x-1) - 7\log(x-2) + \frac{13}{2}\log(x-3) .$$

Homework

As we had a wellness day break, this homework mixes problems from unit 25 and 26.

Problem 26.1: Do without the tabular method:
a) $\int xe^{7x+1}\, dx$.
b) $\int x^2 \log(x)\, dx$.

Problem 26.2: $\int (x-1)^7 \sin(3x)\, dx$. Do this with the tabular method.

Problem 26.3: a) $\int \cos(3x)\sin(5x)\, dx$ (Merry go round).
b) $\int \frac{1}{x^2-14x+45}\, dx$
c) $\int \frac{2}{x^2-9}\, dx$

Problem 26.4: $\int \frac{x^3-x+1}{x^2-1}\, dx$. Subtract first a polynomial.

Problem 26.5: $\int \frac{1}{(x+1)(x-1)(x+7)(x-3)}\, dx$.

Unit 27: Numerical integration

Lecture

27.1. We look here at numerical techniques for computing integrals. Some are variations of basic Riemann sums but they allow speed up or adjust the computation to more complex situations. Johannes Kepler already knew the Simpson rule for one interval. It is also known as the **Kepler Fassregel** as Kepler was able to estimate the content the volume of wine barrel as the height h times an average of the cross sections A, B at both ends and the center C. Kepler saw in 1615 that the volume is close to $h(A+4C+B)/6$, which is the Simpson method. He noticed in his work **Nova Stereometria doliorum vinariorum** that the formula gives even exact results for pyramides, sphere, elliptical paraboloids or hyperboloids.

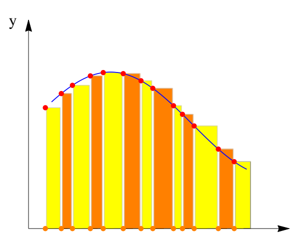
Riemann sum with nonuniform spacing

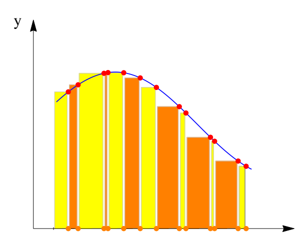
Definition: A more general Riemann sum is obtained by choosing n points $\{x_j\}$ in $[a,b]$ and then sum up. Define $\Delta x_j = x_{j+1} - x_j$:

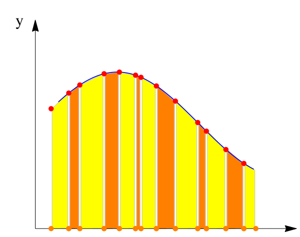
$$S_n = \sum f(x_j)(x_{j+1} - x_j) = \sum_{y_j} f(x_j)\Delta x_j \ .$$

27.2. This flexibility Riemann allows to elastically adapt the mesh size where the function needs more attention. The function $f(x) = \sin(1/(x^2+0.1))$ for example fluctuates near the origin so that more division points are needed near 0.

Definition: For a fixed division $x_0, \dots, x_n$, the sum $L = \sum_{j=0}^{n-1} f(x_j)\Delta x_j$ is called the **left Riemann sum**, the sum $R = \sum_{j=0}^{n-1} f(x_{j+1})\Delta x_j$ the **right Riemann sum**.







If $x_0 = a, x_n = b$ and $\max_j \Delta x_j \to 0$ for $n \to \infty$ then S_n converges to $\int_a^b f(x)\,dx$.

Example: If $x_j - x_k = 1/n$ and $z_j = x_j$, then we have the Archimedes sum defined earlier in this course.

Example: You numerically integrate $\sin(x)$ on $[0, \pi/2]$ with a Riemann sum. Compare the left Riemann sum or the right Riemann sum with the integral itself. In the second case, look at the interval $[\pi/2, \pi]$. **Solution:** you see that in the first case, the left Riemann sum is smaller than the actual integral. In the second case, the left Riemann sum is larger than the actual integral.

Trapezoid rule

Definition: The average $T = (L+R)/2$ between the left and right hand Riemann sum is called the **Trapezoid rule**. Geometrically, it sums up areas of trapezoids instead of rectangles.

27.3. The trapezoid rule does not change things much as it sums up almost the same sum. For the interval $[0, 1]$ for example, with $x_k = k/n$ we have

$$R - L = \frac{1}{n}[f(1) - f(0)] .$$

Simpson rule

Definition: The **Simpson rule** computes the sum

$$S_n = \frac{1}{6n} \sum_{k=1}^{n} [f(x_k) + 4f(y_k) + f(x_{k+1})] ,$$

where $y_k = (x_k + x_{k+1})/2$ is the midpoint between x_k and x_{k+1}.

27.4. The Simpson rule gives the actual integral for quadratic functions: for $f(x) = ax^2 + bx + c$, the formula

$$\frac{1}{v-u} \int_u^v f(x)\,dx = [f(u) + 4f((u+v)/2) + f(v)]/6$$

holds exactly. To prove it just run the following two lines in Mathematica: (== means "is equal")

```
f[x_] := a x^2 + b x + c;
(f[u]+f[v]+4f[(u+v)/2])/6==Integrate[f[x],{x,u,v}]/(v-u)]
Simplify[%]
```

27.5. With a bit more calculus one can show that if f is 4 times differentiable then the Simpson rule is n^{-4} close to the actual integral. For 100 division points, this can give accuracy to 10^{-8} already.

There are other variants which are a bit better but need more function values. If x_k, y_k, z_k, x_{k+1} are equally spaced, then

Definition: The **Simpson 3/8 rule** computes

$$\frac{1}{8n}\sum_{k=1}^{n}[f(x_k)+3f(y_k)+3f(z_k)+f(x_{k+1})] .$$

This formula is again exact for quadratic functions: for $f(x)=ax^2+bx+c$, the formula

$$\frac{1}{v-u}\int_u^v f(x)\,dx = [f(u)+3f((2u+v)/3)+3f((u+2v)/3)+f(v)]/6$$

holds. Just run the two Mathematica lines to check this:

```
f[x_] := a x^2 + b x + c; L=Integrate[f[x],{x,u,v}]/(v-u);
Simplify[(f[u]+f[v]+3f[(2u+v)/3]+3f[(u+2v)/3])/8==L]
```

This **Simpson** 3/8 **method** can be slightly better than the first Simpson rule.

Monte Carlo Method

27.6. A powerful integration method is to chose n random points x_k in $[a,b]$ and look at the sum divided by n. Because it uses randomness, it is called **Monte Carlo method**.

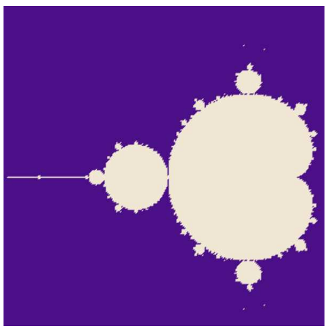
Definition: The **Monte Carlo** integral is the limit S_n to infinity

$$S_n = \frac{(b-a)}{n}\sum_{k=1}^{n} f(x_k) ,$$

where x_k are n random values in $[a,b]$.

27.7. The law of large numbers in probability shows that the **Monte Carlo integral** is equivalent to the **Lebesgue integral** which is more powerful than the Riemann integral. Monte Carlo integration is interesting especially if the function is complicated.

Example: The following two lines evaluate the **area of the Mandelbrot fractal** using Monte Carlo integration. The function F is equal to 1, if the parameter value c of the quadratic map $z \to z^2+c$ is in the Mandelbrot set and 0 else. It shoots $100'000$ random points and counts what fraction of the square of area 9 is covered by the set. Numerical experiments give values close to the actual value around 1.51.... One could use more points to get more accurate estimates.



```
F[c_]:=Block[{z=c,u=1},Do[z=N[z^2+c];If[Abs[z]>3,u=0;z=3],{99}];u];
M=10^5; Sum[F[-2.5+3 Random[]+I(-1.5+3 Random[])],{M}]*(9.0/M)
```

Homework

Problem 27.1: Use the generalized left Riemann sum with $x_0 = 0, x_1 = \pi/6$, $x_2 = \pi/2$, $x_3 = 2\pi/3$ and $x_4 = \pi$ to compute the integral

$$\int_0^\pi 5\sin(x)\, dx$$

without a computer.

Problem 27.2: Use a computer to generate 10 random numbers x_k in $[0,1]$. If you do not have a computer to do that for you, make up some random numbers on your own. Try to be as random as possible. Sum up the cubes x_k^3 of these numbers and divide by 10. Compare your result with $\int_0^1 x^3\, dx$.

Remark. If using a program, increase the value of n as large as you can. Here is a Mathematica code:

```
n=10; Sum[ Random[]^3,{n}]/n
```

Problem 27.3: Use the Simpson rule to compute $\int_0^\pi 5\sin(x)\, dx$ using $n = 2$ intervals $[a,b] = [0,\pi/2]$ or $[a,b] = [\pi/2,\pi]$. On each of these two intervals $[a,b]$, compute the Simpson value

$$\frac{[f(a) + 4f((a+b)/2) + f(b)]}{6}(b-a)$$

with $f(x) = 5\sin(x)$ then add up. Compare with the actual integral.

Problem 27.4: Now use the 3/8-Simpson rule to estimate $\int_0^\pi 5\sin(x)\, dx$ using $n = 1$ intervals $[0,\pi]$. Again compare with the actual integral.

Problem 27.5: a) Use a computer to numerically integrate

$$\int_0^1 \sin(\frac{1}{x^2})\frac{1}{x^2}\, dx\ .$$

b) Do the same with

$$\int_{-1}^1 \sin^2(x)\frac{1}{x^2}\, dx\ .$$

Unit 28: Review

Scope

The second midterm on Friday covers units 16-26: Newton method, catastrophes, integrals, fundamental theorem of calculus and anti-derivatives, areas, volumes, improper integrals, probability distributions, substitution, integration by parts and partial fractions.

Main points

Important integration techniques: **use substitution**, (write down u, du, dx) **use integration by parts** ($\int udv = uv - \int vdu$).

Newton step $T(x) = x - f(x)/f'(x)$.

Catastrophes are parameter values where the number of minima changes. To find the parameter, look where $f''(x_c)$ becomes zero at the critical point x_c.

Definite integrals $F(x) = \int_0^x f(t)\, dt$ were defined as a limit of Riemann sums.

A function $F(x)$ satisfying $F' = f$ is called the anti-derivative of f. The general anti-derivative is $F + c$ where c is a constant.

The **fundamental theorem of calculus** tells $d/dx \int_0^x f(x)\, dx = f(x)$ and $\int_0^x f'(x)\, dx = f(x) - f(0)$.

The integral $\int_a^b g(x) - f(x)\ dx$ is the **signed area between the graphs** of f and g. Places, where $f < g$ are counted negative. When area is asked, split things up.

The integral $\int_a^b A(x)\ dx$ is a **volume** if $A(x)$ is the area of a slice of the solid perpendicular to a point x on an axes.

Write **improper integrals** as limits of definite integrals $\int_1^\infty f(x)\ dx = \lim_{R\to\infty} \int_1^R f(x)\ dx$. We similarly treat points, where f is discontinuous.

One can use CDF's F to compute probabilities: $F(b) - F(a) = \int_a^b f(x)\ dx$.

One can compute **area**, **volume**, **cumulative distribution functions** using integrals.

To determine the **catastrophes** for a family $f_c(x)$ of functions, determine the critical points in dependence of c and find values c, where a critical point changes from a local minimum to a local maximum.

Important integrals

Which one is the derivative which the integral?

$\sin(x)$	$-\cos(x)$.	$\log(x)$	$x\log(x) - x$
$\tan(x)$	$1/\cos^2(x)$.	$1/x$	$\log(x)$
$\arctan(x)$	$1/(1+x^2)$.	$-1/(1+x^2)$	$\text{arccot}(x)$
$1/\sqrt{1-x^2}$	$\arcsin(x)$	$-1/\sqrt{1-x^2}$	$\arccos(x)$

Integration techniques

1) Direct with the FTC
2) Substitution
3) Integration by parts
4) Partial fractions

Improper integrals

$\int_1^\infty 1/x^2\ dx$ Prototype of improper integral which exists.
$\int_1^\infty 1/x\ dx$ Prototype of improper integral which does not exist.

$\int_0^1 1/x \, dx$ Prototype of improper integral which does not exist.
$\int_0^1 1/\sqrt{x} \, dx$ Prototype of improper integral which does exist.

The fundamental theorem

$\frac{d}{dx} \int_0^x f(t) \, dt = f(x)$

$\int_0^x f'(t) \, dt = f(x) - f(0)$.

This implies

$\int_a^b f'(x) \, dx = f(b) - f(a)$

Without limits of integration, we call $\int f(x) \, dx$ the **anti derivative**. It is defined up to a constant. For example $\int \sin(x) \, dx = -\cos(x) + C$.

PDF and CDF

Calculus applies directly if there are situations where one quantity is the derivative of the other.

function	anti derivative
probability density function	cumulative distribution function

Most important integrals

The most important integral is the integral

$$\boxed{\int x^n \, dx = \frac{x^{n+1}}{n+1}}$$

holds for all n different from 1.

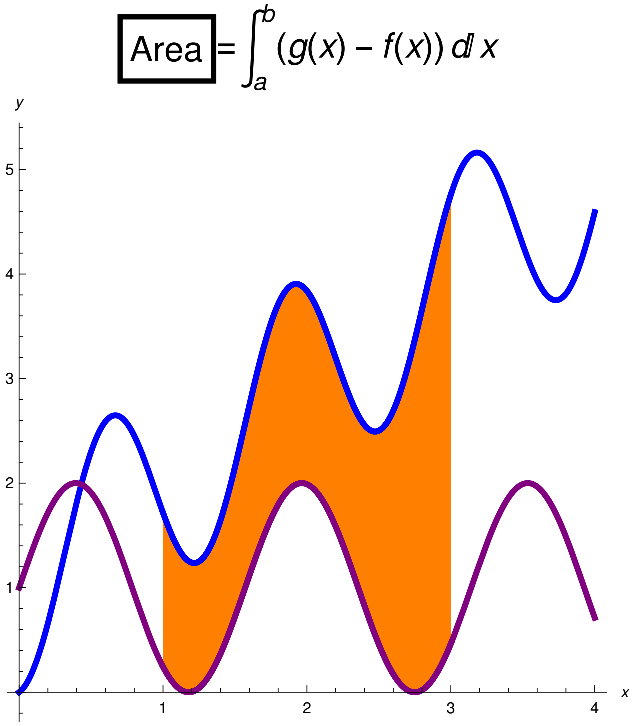
$$\boxed{\int \frac{1}{x} \, dx = \log(x)}$$

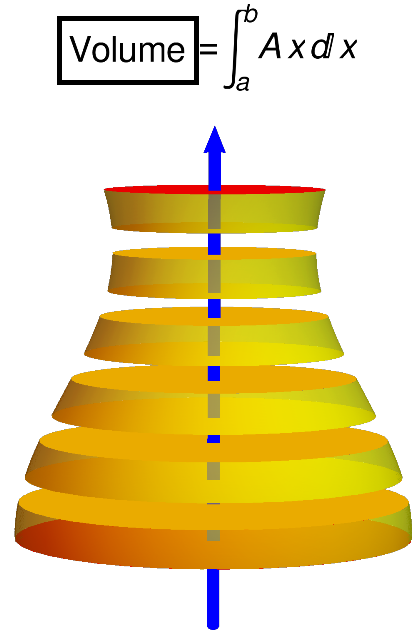
Example: $\int \sqrt{x+7} \, dx = \frac{2}{3}(x+7)^{3/2}$.

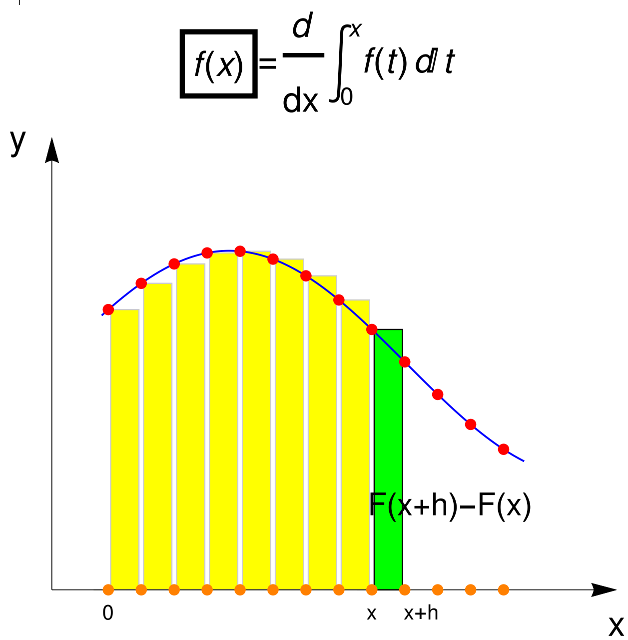
Example: $\int \frac{1}{x+5} \, dx = \log(x+5)$

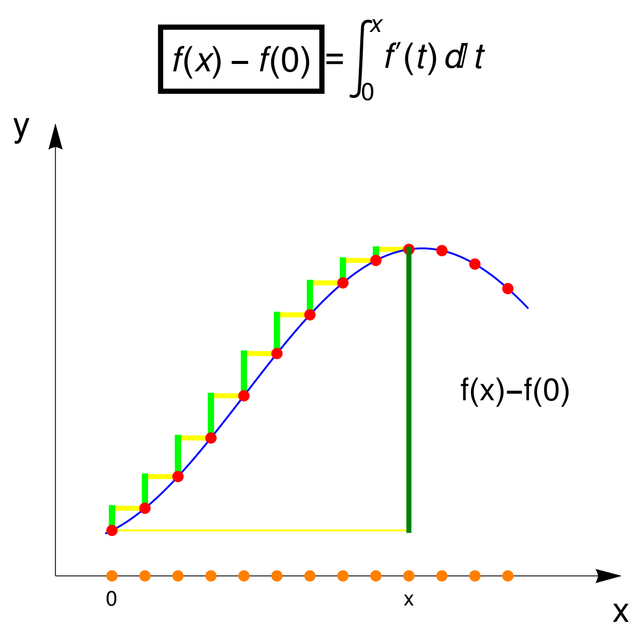
Example: $\int \frac{1}{4x+3} \, dx = \log(4x+3)/4$

Key pictures









Make a picture, whenever we deal with an area or volume computation! For volume computations, just integrate area of the cross section $A(x)$. For area computations integrate $g(x) - f(x)$, where $g(x)$ is above $f(x)$.

Oliver Knill, knill@math.harvard.edu, Math 1a, Harvard College, Spring 2021

INTRODUCTION TO CALCULUS

MATH 1A

Unit 29: Trig Substitution

Lecture

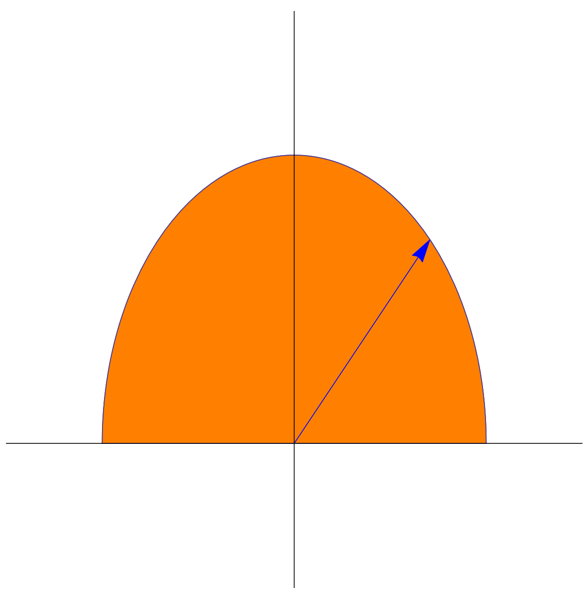
29.1. A **trig substitution** is a substitution, where x is a trigonometric function of u or u is a trigonometric function of x. Here is an important example:

Example: The area of a half circle of radius 1 is given by the integral

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx \; .$$

Solution. Write $x = \sin(u)$ so that $\cos(u) = \sqrt{1-x^2}$. $dx = \cos(u)du$. We have $\sin(-\pi/2) = -1$ and $\sin(\pi/2) = 1$ the answer is

$$\int_{-\pi/2}^{\pi/2} \cos(u)\cos(u) du = \int_{-\pi/2}^{\pi/2} (1+\cos(2u))/2 = \frac{\pi}{2} \; .$$



29.2. Let us do the same computation for a general radius r:

Example: Compute the area of a half disc of radius r which is given by the integral

$$\int_{-r}^{r} \sqrt{r^2-x^2}\, dx \; .$$

Solution. Write $x = r\sin(u)$ so that $r\cos(u) = \sqrt{r^2 - x^2}$ and $dx = r\cos(u)\ du$ and $r\sin(-\pi/2) = -r$ and and $r\sin(\pi/2) = r$. The answer is

$$\int_{-\pi/2}^{\pi/2} r^2 \cos^2(u)\ du = r^2\pi/2\ .$$

29.3. Here is an example, we know already how to integrate. But now we derive it from scratch:

Example: Find the integral

$$\int \frac{dx}{\sqrt{1-x^2}}\ .$$

We know the answer is $\arcsin(x)$. How can we do that without knowing? **Solution.** We can do it also with a trig substitution. Try $x = \sin(u)$ to get $dx = \cos(u)\ du$ and so

$$\int \frac{\cos(u)\ du}{\cos(u)} = u = \arcsin(x) + C\ .$$

29.4. In the next example, $x = \tan(u)$ works. You have to be told that first as it is hard to come up with the idea:

Example: Find the following integral:

$$\int \frac{dx}{x^2\sqrt{1+x^2}}$$

by using the substitution $x = \tan(u)$. **Solution.** Then $1 + x^2 = 1/\cos^2(u)$ and $dx = du/\cos^2(u)$. We get

$$\int \frac{du}{\cos^2(u)\tan^2(u)(1/\cos(u))} = \int \frac{\cos(u)}{\sin^2(u)}\ du = -1/\sin(u) = -1/\sin(\arctan(x))\ .$$

29.5. For trig substitution, the following basic trig identity is important:

$\cos^2(u) + \sin^2(u) = 1$

Depending on whether dividing by $\sin^2(u)$ or $\cos^2(u)$, we get

$1 + \tan^2(u) = 1/\cos^2(u)$, $1 + \cot^2(u) = 1/\sin^2(u)$

These identities come handy: lets look at more examples:

Example: Evaluate the following integral

$$\int x^2/\sqrt{1-x^2}\ dx\ .$$

Solution: Substitute $x = \cos(u), dx = -\sin(u)\ du$ and get

$$\int -\frac{\cos^2(u)}{\sin(u)}\ \sin(u)du = -\int \cos^2(u)\ du = -\frac{u}{2} - \frac{\sin(2u)}{4} + C = -\frac{\arcsin(x)}{2} + \frac{\sin(2\arcsin(x))}{4} + C\ .$$

Example: Evaluate the integral

$$\int \frac{dx}{(1+x^2)^2} .$$

Solution: we make the substitution $x = \tan(u), dx = du/(\cos^2(u))$. Since $1 + x^2 = \cos^{-2}(u)$ we have

$$\int \frac{dx}{(1+x^2)^2} = \int \cos^2(u)\, du = (u/2) + \frac{\sin(2u)}{4} + C = \frac{\arctan(u)}{2} + \frac{\sin(2\arctan(u))}{4} + C .$$

29.6. Here is an other prototype problem:

Example: Find the anti derivative of $1/\sin(x)$. **Solution**: We use the substitution $u = \tan(x/2)$ which gives $x = 2\arctan(u), dx = 2du/(1+u^2)$. Because $1 + u^2 = 1/\cos^2(x/2)$ we have

$$\frac{2u}{1+u^2} = 2\tan(x/2)\cos^2(x/2) = 2\sin(x/2)\cos(x/2) = \sin(x) .$$

Plug this into the integral

$$\int \frac{1}{\sin(x)}\, dx = \int \frac{1+u^2}{2u} \frac{2du}{1+u^2} = \int \frac{1}{u}\, du = \log(u) + C = \log(\tan(\frac{x}{2})) + C .$$

Unlike before, where x is a trig function of u, now u is a trig function of x. This example shows that the substitution $u = \tan(x/2)$ is magic. It leads to the following formulas. We can call it the **magic box**:

$u = \tan(x/2)$
$\boxed{1}$ $dx = \frac{2du}{(1+u^2)}$
$\boxed{2}$ $\sin(x) = \frac{2u}{1+u^2}$
$\boxed{3}$ $\cos(x) = \frac{1-u^2}{1+u^2}$

29.7. The magic box allows us to reduce any rational function involving trig functions to rational functions. We can also let the machine do it: [1]

```
u=Tan[x/2]; Simplify[D[u,x]==(1+u^2)/2]
Simplify[Cos[x]== (1-u^2)/(1+u^2)]
Simplify[Sin[x]== 2u/(1+u^2)]
```

Any function $p(x)/q(x)$ where p, q are trigonometric polynomials can now be integrated using elementary functions.

[1]As human: $\boxed{1}$ differentiate to get $du = dx/(2\cos^2(x/2)) = dx(1+u^2)/2$. $\boxed{2}$ use double angle $\sin(x) = 2\tan(x/2)\cos^2(x/2)$ and then $1/\cos^2(x/2) = 1 + \tan^2(x/2)$. $\boxed{3}$ use double angle $\cos(x) = \cos^2(x/2) - \sin^2(x/2) = (1 - \sin^2(x/2)/\cos^2(x/2))\cos^2(x/2)$ and again $1/\cos^2(x/2) = 1 + \tan^2(x/2)$.

29.8. It is usually a lot of work, but here is an example:

Example: To find the integral

$$\int \frac{\cos(x) + \tan(x)}{\sin(x) + \cot(x)} \, dx$$

for example, we replace $dx, \sin(x), \cos(x), \tan(x) = \sin(x)/\cos(x), \cot(x) = \cos(x)/\sin(x)$ with the above formulas we get a rational expression which involves u only This gives us an integral $\int p(u)/q(u) \, du$ with polynomials p, q. In our case, this would simplify to

$$\int \frac{2u\left(u^4 + 2u^3 - 2u^2 + 2u + 1\right)}{(u-1)(u+1)\left(u^2+1\right)\left(u^4 - 4u^2 - 1\right)} du$$

The method of partial fractions provides us then with the solution.

Homework

Problem 28.1: Find the anti-derivative:

$$\int \sqrt{1 - 16x^2} \, dx \, .$$

Problem 28.2: Find the anti-derivative:

$$\int (1 - x^2)^{3/2} \, dx \, .$$

Problem 28.3: Find the anti-derivative:

$$\int \frac{\sqrt{1 - x^2}}{x^2} \, dx \, .$$

Problem 28.4: Integrate

$$\int \frac{dx}{1 + \sin(x)} \, .$$

Use the substitution $u = \tan(x/2)$ and use the magic box.

Problem 28.5: Use the magic.

a) Compute

$$\int \frac{\tan(x/2) dx}{\cos(x)} \, .$$

b) Do it in the same way:

$$\int \frac{\tan^2(x/2) dx}{\cos(x)} \, .$$

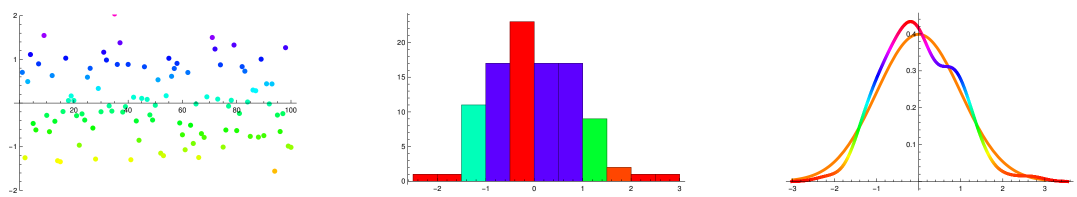
INTRODUCTION TO CALCULUS

MATH 1A

Unit 30: Statistics

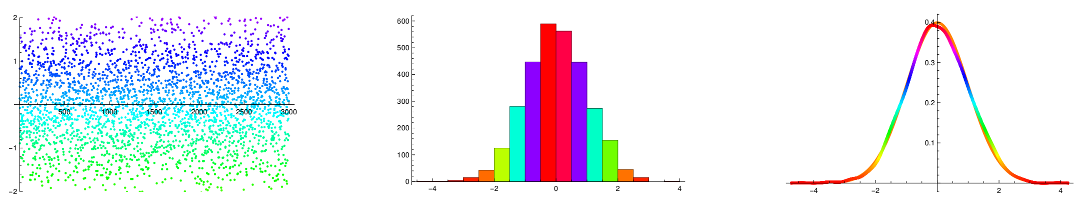
Functions

30.1. Statistics describes the distribution of data using functions. Lets look at the following 100 data points. If we count, how many data values fall into a specific interval, we get a **histogram**. Smoothing this histogram and scaling so that the total integral is 1 produces a **probability distribution function**. This allows us to describe data, discrete sets of points with functions.



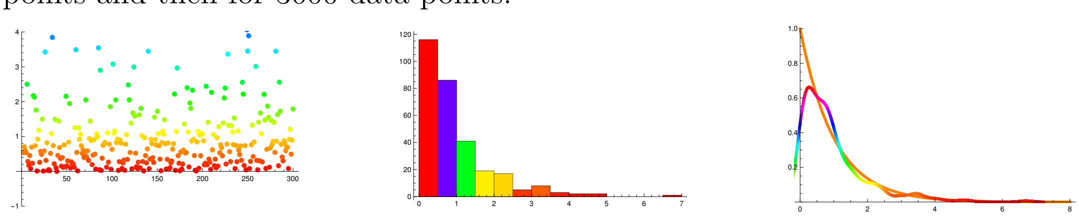
100 data points, the histogram and a smooth interpolation PDF.

30.2. Let us now look at 3000 points. The data distribution has a bell curve shape. In the last picture, we also included the graph of $f(x) = e^{-x^2/2}/\sqrt{2\pi}$.

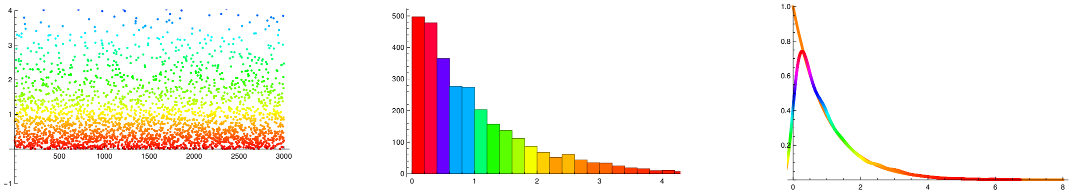


3000 data points, the histogram and a smooth interpolation PDF.

30.3. There are some data of "waiting times. These data are positive. We then draw the histogram and a smooth interpolation function. Lets do it again first for 300 data points and then for 3000 data points.



300 data points, the histogram and a smooth interpolation PDF



3000 other data points, the histogram and a smooth interpolation PDF

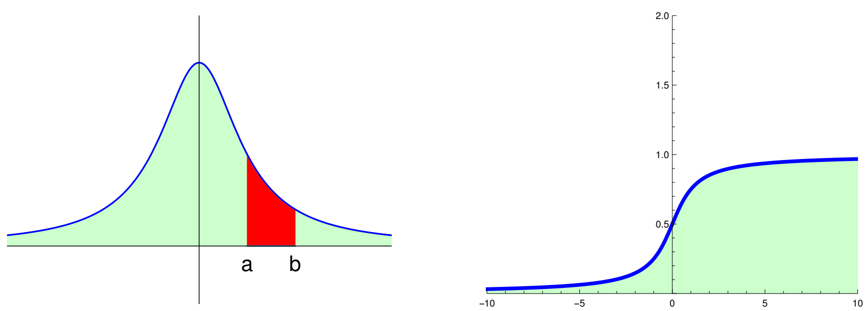
Integration

30.4. We have already defined the probability density function f called PDF and its anti-derivative $F(x)$, the cumulative distribution function CDF.

Definition: Recall that a probability density function is a piecewise continuous function f satisfying $\int_{-\infty}^{\infty} f(x)\, dx = 1$ and which is ≥ 0 everywhere.

Definition: Of great interest are **moments** of the PDF. These are integrals of the form

$$M_n = \int_{-\infty}^{\infty} x^n f(x)\, dx$$



PDF and CDF

30.5. For $n = 0$, we know the answer is always $M_0 = 1$. The first moment M_1 is the expectation or average:

Definition: The **expectation** of probability density function f is

$$m = \int_{-\infty}^{\infty} x f(x)\, dx \; .$$

30.6. The second moment allows us to get the **variance** which is of great importance:

Definition: The **variance** of probability density function f is

$$\text{Var}(f) = \int_{-\infty}^{\infty} x^2 f(x)\, dx - m^2 \; ,$$

where m is the expectation. We can write $\text{Var}(f) = M_2 - M_1^2$.

Definition: More generally, one can look at $\mu_k = \int_{-\infty}^{\infty} (x-m)^k f(x)\, dx$ which is called the k'th **central moment**. We have $\mu_2 = \text{Var}$.

Definition: The square root σ of the variance is called the **standard deviation**.

30.7. The standard deviation tells us what deviation we expect from the mean. From it, one can get the **normalized central moment** $C_k = \mu_k/\sigma^k$ which is $C_k = \int_{-\infty}^{\infty} \frac{(x-m)}{\sigma})^k f(x)\, dx$. Computing moments, central moments and normalized central moments leads often to "integration by parts" problems:

Example: The expectation of the geometric distribution $f(x) = e^{-x}$

$$\int xe^{-x}\, dx = 1 \, .$$

Example: The variance of the geometric distribution $f(x) = e^{-x}$ is 1 and the standard deviation 1 too. To see this, let us compute

$$\int x^2 e^{-x}\, dx \, .$$

x^2	e^{-x}	
$2x$	$-e^{-x}$	$\oplus$
2	e^{-x}	$\ominus$
0	e^{-x}	$\oplus$

Example: You have already computed the expectation of the standard Normal distribution $f(x) = (2\pi)^{-1/2} e^{-x^2/2}$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{-x^2/2}\, dx = 0 \, .$$

Example: The variance of the standard Normal distribution $f(x)$ is $\frac{1}{\sqrt{2\pi}}$ times

$$\int_{-\infty}^{\infty} x^2 e^{-x^2/2}\, dx \, .$$

We can compute this integral by partial integration too but we have to split it as $u = x$ and $v = xe^{-x^2/2}$.

$$-xe^{-x^2/2}|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi} \, .$$

The variance therefore is $\boxed{1}$.

30.8. The next example is for trig substitution:

Example: The distribution supported on $[-1, 1]$ with function $(1/\pi)(1-x^2)^{-1/2}$ there and 0 everywhere else is called the **arcsin-distribution**. What is the cumulative distribution function? What is the mean m? What is the standard deviation σ? We will compute this in class. The answers are $m = 0, \sigma = 1/\sqrt{2}$.

Homework

Problem 30.1: The function $f(x) = \cos(x)/2$ on $[-\pi/2, \pi/2]$ is a probability density function. Its mean is 0. Find its variance $\int_{-\pi/2}^{\pi/2} x^2 \cos(x)\, dx$.

Problem 30.2: The **uniform distribution on** $[a, b]$ is a distribution with probability density function is $f(x) = 1/(b-a)$ for $a \leq x \leq b$ and 0 elsewhere. Let $a = 1$ and $b = 5$;
a) Find the n'th moment $M_n = \int_{-\infty}^{\infty} x^n f(x)\, dx$ in general.
b) Now compute the variance $\text{Var}[f] = M_2 - M_1^2$ and the standard deviation $\sigma = \sqrt{\text{Var}[f]}$.

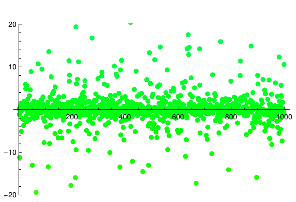
Problem 30.3: Define $f(x)$ to be 0 for $x < 0$ and for $x > 0$ to be

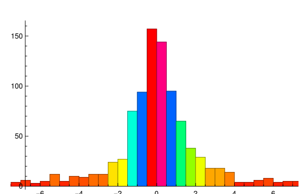
$$f(x) = \frac{1}{\log(2)} \frac{e^{-x}}{1 + e^{-x}} .$$

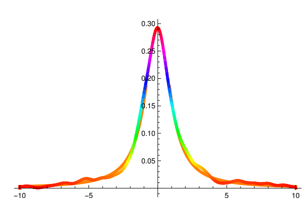
a) Find the CDF F and verify that f is a probability density function. (Note that $F(x) = 0$ for $x \leq 0$ and especially $F(0) = 0$.)
b) Use a computer to numerically compute the expectation $m = M_1$.
c) Use a computer to compute the second moment M_2.
d) What is the standard deviation $\sigma = \sqrt{M_2 - M_1^2}$ of the distribution?
P.S. Your computer algebra system might tell you $M_1 = \zeta(2)/2, M_2 = 3\zeta(3)/2$. In general $M_n = \zeta(1+n)n!(2^n - 1)/2^n$ for the **zeta function**.

Problem 30.4: a) Verify again that the **Cauchy distribution** with PDF $f(x) = \frac{(1/\pi)}{x^2+1}$ has the CDF $F(y) = 1/2 + \arctan(y)/\pi$.
b) What can you say about the variance of this distribution?

Problem 30.5: Let us quickly verify that if we take random numbers x in $[0, 1]$ then the data $\tan(x)$ are Cauchy distributed: just check that the probability that $y = \tan(\pi x)$ is in $[a, b]$ is $F(b) - F(a)$ for the Cauchy CDF appearing in the last problem. Now, use a calculator and compute 10 random Cauchy distributed numbers. In Mathematica such numbers can be accessed by Tan[Pi ∗ Random[]].







Unit 31: Calculus and Economics

Lecture

31.1. Calculus plays a pivotal role in **economics**. This unit is an opportunity to review extrema problems and get acquainted with some jargon in economics. Economists talk differently: $f' > 0$ means **growth** or **boom**, $f' < 0$ means **decline** or **recession**, a vertical asymptote is a **crash**, a horizontal asymptote is a **stagnation**, a discontinuity is **"inelastic behavior"**, the derivative of something is the **"marginal"** of it. An example is marginal revenue.

Definition: The **marginal cost** is the derivative of the **total cost**.

31.2. The marginal cost and total cost are functions of the quantity x of goods:

Example: Assume the total cost function is $C(x) = 10x - 0.01x^2$. Use the marginal cost in order to minimize the total cost. **Solution**. Differentiate $C' = 10 + 0.02x$ At a minimum, this derivative is zero. Here at $x = 50$.

Example: You sell spring water. The marginal cost to produce it at time x (years) is $f(x) = 1000 - 2000\sin(x/6)$. For which x is the total cost maximal? **Solution**. We look for points where $F'(x) = f(x) = 0, F''(x) < 0$. This is the case for $x = \pi$. The cost is maximal in about 3 years.

Example: In the book "Don't worry about Micro, 2008", by Dominik Heckner and Tobias Kretschmer, the following strawberry story appears: (italics is verbatim):

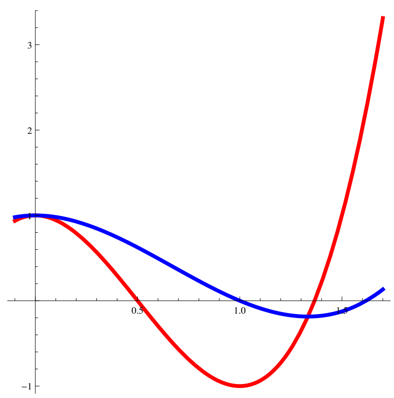
Suppose you have all sizes of strawberries, from very large to very small. Each size of strawberry exists twice except for the smallest, of which you only have one. Let us also say that you line these strawberries up from very large to very small, then to very large again. You take one strawberry after another and place them on a scale that sells you the average weight of all strawberries. The first strawberry that you place in the bucket is very large, while every subsequent one will be smaller until you reach the smallest one. Because of the literal weight of the heavier ones, average weight is larger than marginal weight. Average weight still decreases, although less steeply than marginal weight. Once you reach the smallest strawberry, every subsequent strawberry will be larger which means that the rate of decease of the average weight becomes smaller and smaller until eventually, it stands still. At this point the marginal weight is just equal to the average weight.

31.3. If $F(x)$ is the **total cost function** in dependence of the quantity x, then $F' = f$ is called the **marginal cost**.

Definition: The function $g(x) = F(x)/x$ is called the **average cost**.

Definition: A point where $f = g$ is called a **break-even point**.

Example: If $f(x) = 4x^3 - 3x^2 + 1$, then $F(x) = x^4 - x^3 + x$ and $g(x) = x^3 - x^2 + 1$. Find the break even point and the points, where the average costs are extremal. **Solution:** To get the break even point, we solve $f - g = 0$. We get $f - g = x^2(3x - 4)$ and see that $x = 0$ and $x = 4/3$ are two break even points. The critical point of g are points where $g'(x) = 3x^2 - 4x$. They agree:



31.4. The following theorem tells that the marginal cost is equal to the average cost if and only if the average cost has a critical point. Since total costs are typically concave up, we usually have "break even points are minima for the average cost". Since the strawberry story illustrates it well, lets call it the "strawberry theorem":

Strawberry theorem: $g'(x) = 0$ if an only if $f = g$.

Proof.

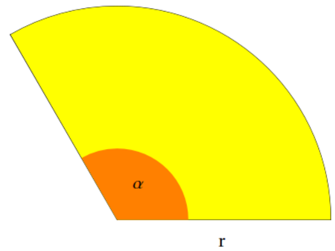
$$g' = (F(x)/x)' = F'/x - F/x^2 = (1/x)(F' - F/x) = (1/x)(f - g) .$$

More extremization

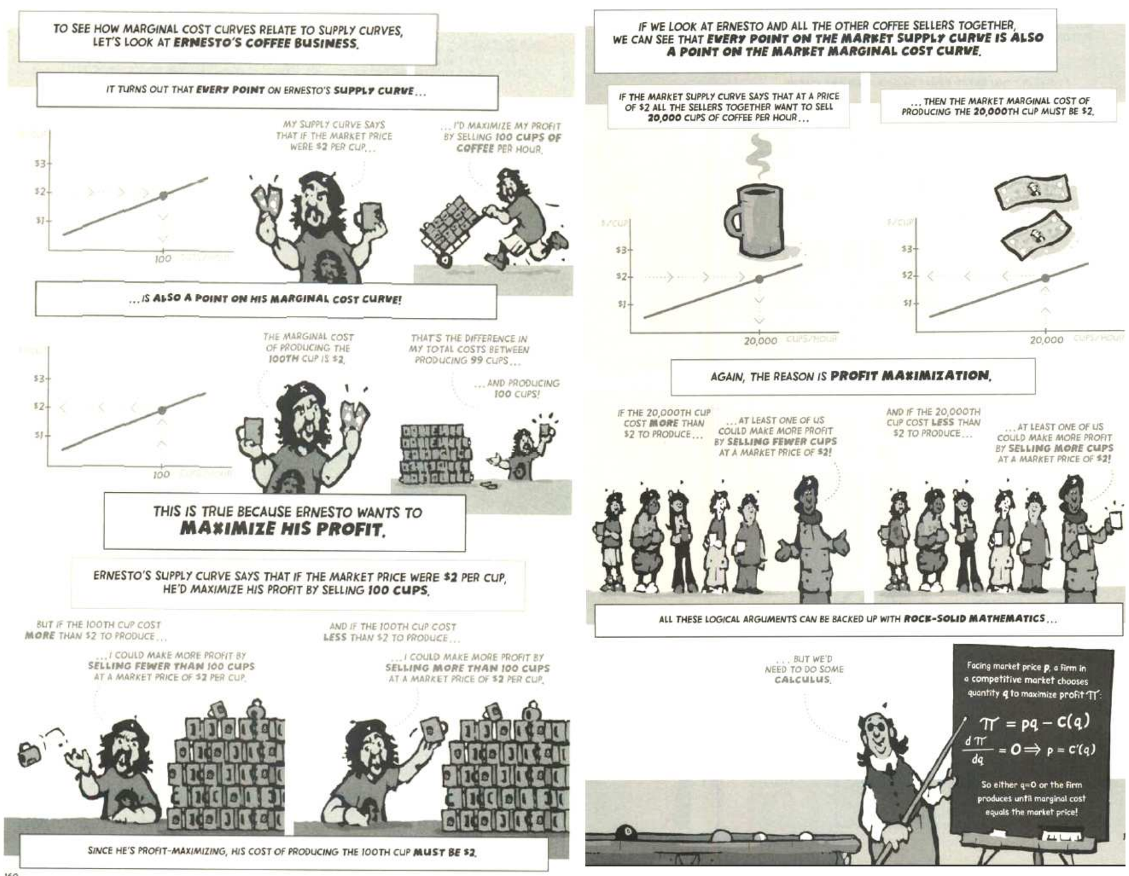
31.5. In the second part of this lecture we still want to look at more extremization problems, also in the context of data.

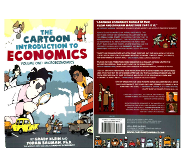
Example: Find the **rhomboid** with side length 1 which has maximal area. Use angle α to extremize. Solution. The area is $base * height = 1 * \sin(\alpha)$. This is maximal at $\pi/2$.

Example: Find the sector of radius$r = 1$ and angle α which has minimal circumference $f = 2r + r\alpha$ if the area $r^2\alpha/2 = 1$ is fixed. **Solution.** Find $\alpha = 2/r^2$ from the second equation and plug it into the first equation. We get $f(r) = 2r + 2/r$. Now the task is to find the places where $f'(r) = 0$.



Example: Find the ellipse of length $2a$ and width $2b$ which has fixed area $\pi ab = \pi$ and for which the sum of diameters $2a + 2b$ is maximal. **Solution.** Find $b = 1/a$ from the first equation and plug into the second equation. Again we have to extremize $f(a) = 2a + 2/a$ as before.





Source: Grady Klein and Yoram Bauman, The Cartoon Introduction to Economics: Volume One Microeconomics, published by Hill and Wang. You can detect the strawberry theorem ($g' = 0$ is equivalent to $f = g$) can be seen on the blackboard.

Homework

Problem 31.1: a) Find the break-even point for an economic system if the marginal cost is $f(x) = 1/x$.
b) Assume the marginal cost is $f(x) = x^7$. Verify that the average cost $g(x) = F(x)/x$ satisfies $8g(x) = f(x)$ and that $x = 0$ is the only break even point.

Problem 31.2: Let $f(x) = \cos(x)$. Compute $F(x)$ and $g(x)$ and verify that $f = g$ agrees with $g' = 0$.

Problem 31.3: For smaller groups, production usually increases when adding more workforce. After some time, bottle necks occur, not all resources can be used at the same management and bureaucracy is added. We make a model to find the maximal production parameters. The **production function** in an office gives the production $Q(L)$ in dependence of labor L. Assume

$$Q(L) = 5000L^3 - 3L^5 .$$

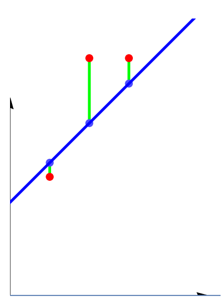
Find L which gives the maximal production.

Problem 31.4: **Marginal revenue** f is the rate of change in **total revenue** F. As total and marginal cost, these are functions of the **cost** x. Assume the total revenue is $F(x) = -5x - x^5 + 9x^3$. Find the points, where the total revenue has a local maximum.

Problem 31.5: We do linear regression. Find the best line $y = mx + b$ through the points

$$(x_1, y_1) = (1, 3), (x_2, y_2) = (2, 6), (x_3, y_3) = (3, 6) .$$

To do so, first center the data by subtracting the average (2,5), then minimize $f(m) = \sum_{k=1}^{3}(mx_k - y_k)^2$ Now $m(x-2)+5$ is the best fit.



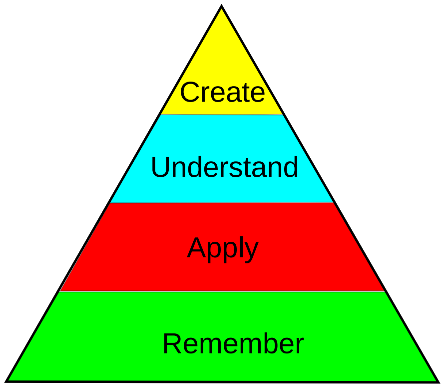
Unit 32: Calculus and AI

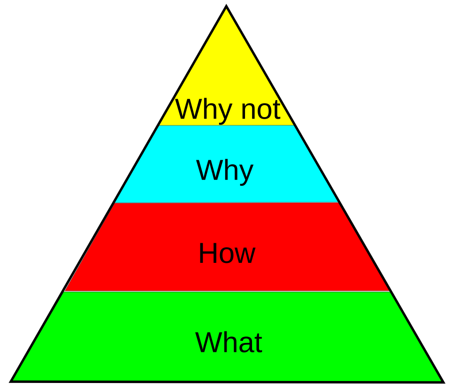
LECTURE

About intelligence

32.1. Artificial intelligence involves also learning aspects and so is interesting for education. In order to build an artificial intelligence entity, one needs to teach it also how to learn. In a project of 2013, we have done some AI programming to build an entity called Sofia. It gave us also insight on **how humans learn**. Teaching definitions for example is simple. A machine is good in memorizing stuff. Also algorithms are no problem to learn for a machine if the task is communicated clearly. What is much more difficult is "teaching insight". How can one bring across to a machine what are "relevant" core principles, what is "important", or what is "good taste"? The hardest part is to teach being creative, or to discover new things. In our project, we had boiled the process of learning and teaching to 4 questions: "What, How, Why and Why not?". We realized later that this is a variant of the **Bloom taxonomy** which splits learning into **factual**, **conceptual**, **procedural** and **metacognitive** components.

32.2. Here is an example in calculus. We teach the concept of derivative. What is a derivative? How does one compute a derivative? Why does one compute derivatives? Why does one not just compute something else? It is no problem to teach a machine that the derivative of $f(x)$ is the limit $(f(x+h)-f(x))/h$ in the limit $h \to 0$. It is also no problem to teach the machine to take the derivative of a function like $\sin(5x)x$. All computer algebra systems know that already. The question why one wants to compute derivatives is harder. But it is teachable too. We want to compute derivatives for example because we want to find maxima or minima of quantities. The question of how we can extend the concept of derivative or replace it with something else is harder. There are other notions in mathematics which have done so. In quantum calculus, one looks at $Df(x) = f(x+1)-f(x)$ for example, in other parts of math, one has notions of "derivations" which formalize operations satisfying the Leibniz rule $(fg)' = f'g+fg'$. In quantum mechanics, the derivative is essentially the "momentum operator" generating translation. To teach a machine to come up with such connections and concepts is much, much harder.





A modified Bloom taxonomy. A major change is to put the "apply" part before the "understanding" part. The rationale is that in almost all situations of learning, one first learns how to do something before knowing why it works.

Generating calculus problems

32.3. We ask "Sofia", our artificial intelligence teacher to automatically build worksheets or exam problems as well as solutions. In order to generate problems, we first must build **random functions**. When asked "give me an example of a function", the system should generate functions of some complexity:

> **Definition:** A **basic function** is a function from the 10 functions $\{\sin, \cos, \log, \exp, \tan, \mathrm{sqt}, \mathrm{pow}, \mathrm{inv}, \mathrm{sca}, \mathrm{tra}\,\}$.

32.4. Here $\mathrm{sqt}(x) = \sqrt{x}$ and $\mathrm{inv}(x) = 1/x^k$ for a random integer k between -1 and -3, $\mathrm{pow}(x) = x^k$ for a random integer k between 2 and 5. $\mathrm{sca}(x) = kx$ is a scalar multiplication for a random nonzero integer k between -3 and 3 and $\mathrm{tra}(x) = x + k$ translates for a random integer k between -4 and 4.

32.5. Second, we use addition, subtraction multiplication, division and composition to build more complicated functions:

> **Definition:** A **basic operation** is an operation from the list $\{f \circ g, f + g, f * g, f/g, f - g\,\}$.

32.6. The operation x^y is not included because it is equivalent to $\exp(x\log(y)) = \exp\circ(x \cdot \log)$. We can now build functions of various complexities:

> **Definition:** A **random function** of complexity n is obtained by taking n random basic functions $f_1, \dots, f_n$, and n random basic operators $\oplus_1, \dots, \oplus_n$ and forming $f_n \oplus_n f_{n-1} \oplus_{n-1} \cdots \oplus_2 f_1 \oplus_1 f_0$ where $f_0(x) = x$ and where we start forming the function from the right.

Example: **Visitor:** "Give me an easy function": Sofia looks for a function of complexity one: like $x\tan(x)$, or $x+\log(x)$, or $-3x^2$, or $x/(x-3)$.

Example: **Visitor:** "Give me a function": Sofia returns a random function of complexity two: $x\sin(x)-\tan(x)$, or $-e^{\sqrt{x}}+\sqrt{x}$ or $x\sin(x)/\log(x)$ or $\tan(x)/x^4$.

Example: **Visitor:** "Give me a difficult function": Sofia builds a random function of complexity four like $x^4e^{-\cos(x)}\cos(x)+\tan(x)$, or $x-\sqrt{x}-e^x+\log(x)+\cos(x)$, or $(1+x)(x\cot(x)-\log(x))/x^2$, or $(-x+\sin(x+3)-3)\csc(x)$

32.7. Now, we can build a random calculus problem. To give you an idea, here are some templates for integration problems:

Definition: A **random integration problem** of complexity n is a sentence from the sentence list { "Integrate $f(x)=F(x)$", "Find the anti derivative of $F(x)$", "What is the integral of $f(x)=F(x)$?", "You know the derivative of a function is $f'(x)=F(x)$. Find $f(x)$." }, where F is a random function of complexity n.

Example: **Visitor** "Give me a differentiation problem". **Sofia:** Differentiate $f(x)=x\sin(x)-\frac{1}{x^2}$. The answer is $\frac{2}{x^3}+\sin(x)+x\cos(x)$.

Example: **Visitor**: "Give me a difficult integration problem". **Sofia**: Find f if $f'(x)=\frac{1}{x}+\big(3\sin^2(x)+\sin(\sin(x))\big)\cos(x)$. The answer is $\log(x)+\sin^3(x)-\cos(\sin(x))$.

Example: **Visitor**: "Give me an easy extremization problem". **Sofia**: Find the extrema of $f(x)=x/\log(x)$. The answer is $x=e$.

Example: **Visitor**: "Give me an extremization problem". **Sofia**: Find the maxima and minima of $f(x)=x-x^4+\log(x)$. The extrema are

$$-\frac{\sqrt{\left(9+\sqrt{3153}\right)^{2/3}-8\sqrt[3]{6}}+\sqrt{8\sqrt[3]{6}-\left(9+\sqrt{3153}\right)^{2/3}\left(1+6\sqrt{\frac{2}{9+\sqrt{3153}-8\sqrt[3]{6\left(9+\sqrt{3153}\right)}}}\right)}}{2 2^{5/6}\sqrt[3]{3}\sqrt[6]{9+\sqrt{3153}}}.$$

The last example shows the perils of random generation. Even so the function had decent complexity, the solution was difficult. Solutions can even be transcendental. This is not a big deal: just generate a new problem. By the way, all the above problems and solutions have been generated by Sofia. The dirty secret of calculus books is that there are maybe a thousand different type of questions which are usually asked. This is a reason why textbooks have become boring clones of each other and companies like "Aleks"," Demidec" etc exist which constantly mine the web and course sites like this and homework databases like "webwork" (now a company edfinity) which contain thousands of pre-compiled problems in which randomness is already built in.

Automated problem generation is the "fast food" of teaching and usually not healthy. But like "fast food" has evolved, we can expect more and more computer assisting in calculus teaching.

Be assured that for this course, problems have been written by hand. (Sometimes Mathematica is used to see whether answers are reasonable). Handmade problems can sometimes a bit "rough" but can be more interesting. Still, you can see a worksheet which has been generated entirely by a program.

Homework

Problem 32.1: We build a differentiation problem by combining log and sin and exp. Differentiate all of the 6 combinations $\log(\sin(\exp(x)))$, $\log(\exp(\sin(x)))$,$\exp(\log(\sin(x)))$, $\exp(\sin(\log(x)))$, $\sin(\log(\exp(x)))$ and $\sin(\exp(\log(x)))$.

Problem 32.2: Four of the 6 combinations of log and sin and exp can be integrated as elementary functions.
a) Find all these cases
b) Do these integrals.

Problem 32.3: From the 10 functions f and 10 functions g and 5 operations, we can build 500 functions. Some can not be integrated. An example is $\exp(\sin(x))$. Find 4 more which can not be integrated by you now by any computer algebra system.

Problem 32.4: One of the most difficult things to teach is creativity. Lets try to be creative. Build an extremization problem which is applied. Here is an example (of course your example should be different than this): **A common theme for extrema are area and length. Invent an extremum problem involving an isoscele triangle. It should be of the form: "Maximize the area ... ". Now solve the problem.**

Problem 32.5: Be creative: a) Create your own area problem for a region in the plane which has not appeared in lecture, homework or exam. b) Create a volume problem for a surface of revolution. The problem should not have appeared yet in lecture, homework or exams. Make the problem so that it can be solved. Now solve your problem.

Oliver Knill, knill@math.harvard.edu, Math 1a, Harvard College, Spring 2021

Lecture 33: Calculus and Music

Music is a function

33.1. Calculus matters in music because a piece of music is just a **function**. If you feed a loudspeaker the function $f(t)$ which leads to a displacement of the membrane, the pressure variations in the air are sound waves which then reach your ear, where your ear drum oscillate allowing you to hear the sound. Plotting and playing works the same way. In Mathematica, we can play a function by replacing "Plot" with "Play". For example:

```
Play[ Sin[2Pi 1000 x^2],{x,0,10}]
```

33.2. While the function f contains all the information about the music piece, the computer needs to store this as **data**. A common data format is the ".WAV" file, which contains sampled values of the function, usually with a sample rate of 44100 readings per second. Since our ear does not hear frequencies larger than 20'000 KHz, a sampling rate of 44.1K is good enough by a **theorem of Nyquist-Shannon**. More sophisticated storage possibilities exist. A ".MP3" file for example encodes the function in a compressed way. To get from the sample values $f(k)$ the function back, the **Whittaker-Shannon interpolation formula**

$$f(t) = \sum_{k=1}^{n} f(k)\text{sinc}(t-k)$$

can be used. It involves the **sinc** function $\text{sinc}(x) = \sin(x)/x$ which we have seen earlier.

The wave form and hull

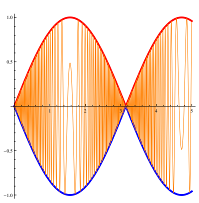
33.3. Periodic signals can serve as **building blocks** of sound. Assume $g(x)$ is a 2π-periodic function, we can generate a sound of 440 Hertz when playing the function $f(x) = g(440 \cdot 2\pi x)$. If the function does not have a smaller period, then we hear the A **tone**. It is a tone with 440 Hertz. We can **modulate** this sound with a **hull function** $h(x)$ and write $f(x) = h(x)g(4402\pi x)$.

Definition: A periodic function g is called a **wave form**.

33.4. The wave form makes up the **timbre** of a sound which allows to model music instruments with macroscopic terms like **attack**, **vibrato**, **coloration**, **noise**, **echo**, **reverberation** and other characteristics.

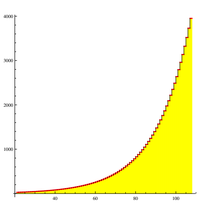
Definition: The **hull function** $h(x)$ is an interpolation of successive local maxima of f.

33.5. For the function $f(x) = \sin(100x)$ for example, the hull function is $h(x) = 1$ For $f(x) = \sin(x)\sin(100x)$ the hull function is $h(x) = |\sin(x)|$. With slight abuse of notation, we sometimes just say $\sin(x)$ is the hull function as the function is sandwiched by the envelopes $\sin(x)$ and $-\sin(x)$.



We can not hear the actual function $f(x)$ because the function changes too fast that we can notice individual vibrations. But we can hear the hull function. We can hear **large scale amplitude** changes like **creshendi** or **diminuendi** or a **vibrato**. When playing two frequencies which are close, one can hear **interference**, the sound analogue of **Moiré patterns** in optics.

The scale



33.6. Western music uses a discrete set of frequencies. This scale is based on the **exponential function**. The frequency f is an exponential function of the scale s. On the other hand, if the frequency is known then the scale number is a logarithm. This is a nice application of the logarithm:

Definition: A frequency f has the **Midi number** $s = 69 + 12 \cdot \log_2(f/440)$. The **piano scale function** or **midi function** gives back $f(s) = 440 \cdot 2^{(s-69)/12}$.

33.7. The Midi tone $s = 100$ for example is a sound of $f = 2637.02$ Hertz (oscillations per second).

The **piano scale function** $f(s) = 440 \cdot 2^{(s-69)/12}$ is an exponential function $f(s) = be^{as}$ which satisfies $f(s + 12) = 2f(s)$.

```
midifrequency[m_] := N[440 2^((m - 69)/12)]
```

33.8. A classical piano has 88 keys which scale from 21 to 108. The frequency ranges from $f = 27.5Hz$, the sub-contra-octave A, to the highest $f = 4186.01Hz$, the 5-line octave C.

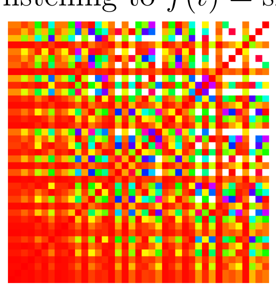
33.9. Filters: a function can be written as a sum of sin and cos functions. Our ear does this so called **Fourier decomposition** automatically. We can so hear melodies, filter out part of the music and hum it.
Pitch and autotune: it is possible to filter out frequencies and adapt their frequency. The popular filter **autotune** moves the frequencies around correcting wrong singing. If 440 Hertz (A) and 523.2 Hertz (C) for example were the only allowed frequencies, the filter would change a function $f(x) = \sin(2\pi 441x) + 4\cos(2\pi 521x)$ to $g(x) = \sin(2\pi 440x) + 4\cos(2\pi 523.2x)$. **Rip and remix**: if f and g are two songs, we can build the average $(f+g)/2$. A composer does this using **tracks**. Different instruments are recorded independently and then mixed together. A guitar $g(t)$, a voice $v(t)$ and a piano $p(t)$ together can form $f(t) = ag(t)+bv(t)+c(p(t)$ with suitably chosen constants a, b, c. **Reverberate and echo**: if f is a song and h is some time interval, we can look at $g(x) = Df(x) = [f(x+h)-f(x)]/h$. For small h, like $h = 1/1000$ the song does not change much because hearing $\sin(kx)$ or $\cos(kx)$ produces the same song. However, for larger h, one can get **reverberate** or **echo** effects.

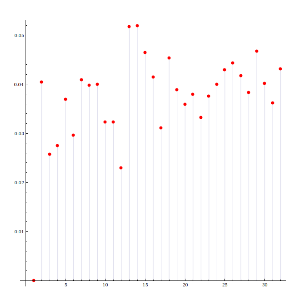
33.10. Mathematics and music have a lot of overlap. Besides wave form analysis and music manipulation operations and symmetry, there are **encoding and compression problems**. A **Diophantine problem** is the question how well a frequency can be approximated by rationals. Why is the **chromatic scale** based on $2^{1/12}$ so effective? **Indian music** for example uses **micro-tones** and a scale of 22. The 12-tone scale has the property that many powers $2^{k/12}$ are close to rational numbers. This can be quantified with the **scale fitness**

$$M(n) = \sum_{k=1}^{n} \mathrm{min}_{\mathrm{p,q}} |2^{k/n} - \frac{p}{q}| G(p/q)$$

where $G(n/m)$ is Euler's **gradus suavitatis** (="degree of sweetness") defined as $G(n/m) = 1 + \sum_{p|n*m}(p-1)$ in which the sum runs over all prime factors p of $n*m$. For example $G(3/4) = 1 + (2-1) + (2-1) + (3-1)$ because $3*4 = 12 = 2*2*3$.

33.11. The figure below illustrates why the 12-tone scale minimizes $M(n)$. We could also replace the concept of octave. Stockhausen experimented with replacing 2 with 5 and used the **Stockhausen scale** $5^{k/25}$. It is $f(t) = \sin(2\pi t 440 \cdot 5^{[t]/25})$, where $[t]$ is the largest integer smaller than t. The familiar **12-tone scale** can be admired by listening to $f(t) = \sin(2\pi t 440 \cdot 2^{[t]/12})$.





Example: The perfect fifth 3/2 has the gradus suavitatis $1 + E(6) = 1 + 2 = 3$ which is the same than the perfect fourth 4/3 for which $1+E(12) = 1+(2-1)(3-1)$. You can listen to the perfect fifth $f(x) = \sin(1000x)+\sin(1500x)$ or the perfect fourth $\sin(1000x)+\sin(1333x)$ and here is a function representing an **accord** with four notes $\sin(1000x) + \sin(1333x) + \sin(1500x) + \sin(2000x)$.

Homework

Problem 33.1: Modulation. Draw and play the following function

$$f(x) = \cos(4000x) - \cos(4011x)$$

for three seconds. You can your AI and just tell it in words what you want to do! How many up and downs to you hear in the hull? Do the same for

$$f(x) = \cos(4000x) - \cos(4021x)\ .$$

Here is how to play a function with Mathematica or Wolfram alpha:

```
Play[Cos[x] Sin[Exp[2 x]]/x, {x, 0,9}]
play  sin(1000 x)
```

Problem 33.2: Amplitude modulation (AM): If you listen to $f(x) = |\cos(x^2)|\sin(1000x)$ you hear an amplitude change. Draw the hull function or listen to it and count how many increases in amplitudes to you hear in 10 seconds.

Problem 33.3: Other tonal scales, Midi number: As a creative musician, we create our own tonal scale. You decide to take the 8'th root of 3 as your basic frequency change from one tone to the next.
a) After how many tonal steps has the frequency f tripled?
b) Build the midi function and then write down the inverse for your tonal scale.

Problem 33.4: a) What is the frequency of the Midi number $s = 22$?
b) Which midi number belongs to the frequency $f = 2060 Herz$?

Problem 33.5: Gradus Suavitatis. a) What is the gradus suavitatis of 49/64? b) What is the gradus suavitatis of 541/221?

INTRODUCTION TO CALCULUS

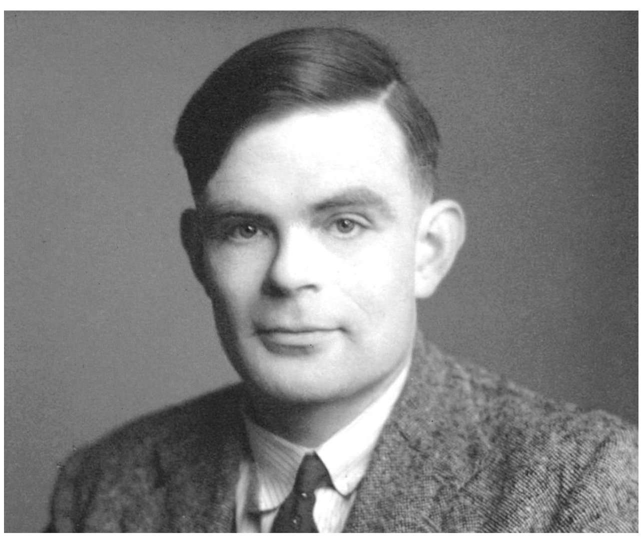
MATH 1A

Unit 34: More Data stories

34.1. In projects you have explored the derivative function of the prime number function $p(n)$ giving the $n'th$ prime, you have computed with functions describing polyhedra, you have seen unpredictability of simple deterministic systems and you have used Monte Carlo simulations to find areas. In this lecture we look at some more data stories.

Turing's picture

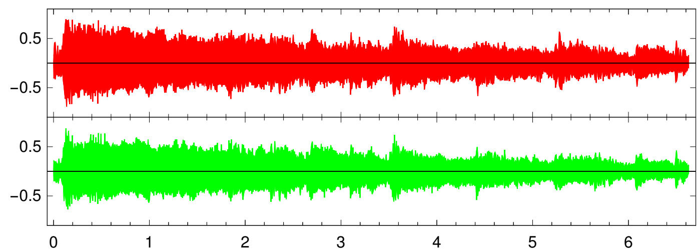
34.2. Information is stored in the form of **data**. Data can always be stored as numerical values $f(k)$, where k is a label. On a computer it is one function $f(1), \cdots, f(n)$ with $f(k)$ taking values in $\{0, \dots, 254\}$ and k is the **memory address** and where where n is the total number of **Bytes** the computer can store. It was **Alan Turing** who illustrated best what **data** and what computing with data means. He coined the concept of a **Turing machine**.



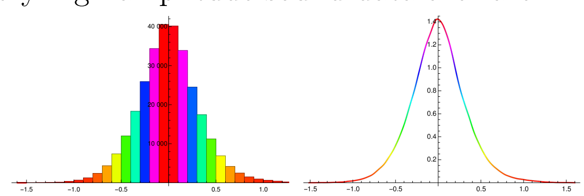
34.3. In that machine, all information is stored on data set $\{...x_{-m},\ x_0, x_n, ...\}$ on which only finitely many 1 appear. This **tape** contains the data. The machine has finitely many states $A_1, ..., A_k$. Depending on whether x_0 is 0 or 1 and what the state of the machine is, the machine now either replaces x_0 with some 0 or 1 or then moves the tape left or right or then changes the state. Turing demonstrated, that all computations we are capable to do can be done also by such a simple machine. The point is that all data we are ever can measure, process or produce can be described by a function $f(k)$ taking values 0 and 1 and having the property that $f(k) = 1$ only for finitely many k.

Audio data

34.4. Of course, these data are organized in a more convenient forms like a file representing a song or an array of numbers encoding a picture or an array of arrays of numbers that encode a movie. The following picture shows a few seconds of audio data of the song "Bohemian Rhapsody". It is the part "Mamaaah, Ohhhh". The command $\{f, g\} = AudioData[Import["queen.wav"]]$ gives you to two discrete functions of length $n = 292669$ corresponding to $n/44100 = 6.6$ seconds of sound. Looking up values like $f(4) = f[[4]]$ gives -0.091 which the amplitude of the left sound channel at time $4/44100$ seconds. The function value $g(4) = g[[4]]$ gives 0.063507, the amplitude of the right sound channel. The function values range from -1 to 1.

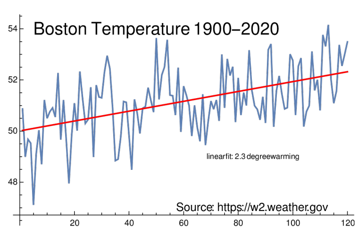


34.5. Anything which has been developed for calculus can be applied to data. For example, we can look for differences $f(k+1) - f(k)$. We can sum up data, average data, produce distributions. For the above sound clip, we can draw the histogram as well as the smooth histogram which is the distribution function of the sound data. What these histograms show is how many sound notes were in amplitude in a given interval. Naturally, very high amplitude sound data are rare.



Whether data

34.6. While multi-variable calculus and linear algebra and probability theory help more effectively to visualize and reduce data, much insight can be gained from data depending on one variable only: we want to know the value $S(n)$ of a stock prize, the temperature $T(n)$ on day n. Here is the development of the average year temperature in Boston over the last 121 years from 1990 to 2020 [1]

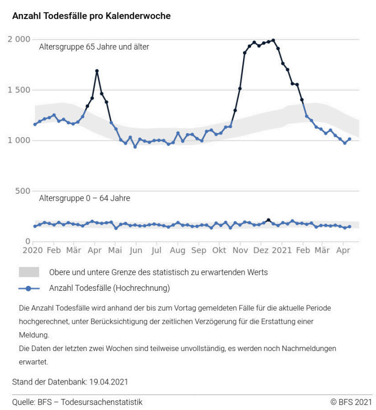


[1] https://w2.weather.gov

Health data

34.7. A good model needs to be able to predict what the data should look like other times. If we build a model we then compare the data with the model. Weather data are very reliable. Everybody can measure the temperature and compare it with the data which are published. Health data are much less reliable. While we have pretty good and accurate data about birth and death of the population, we have much less reliable data about the cause of death.

34.8. Switzerland has quite good and reliable statistical data. The BFS (Bundesamt für Statistik) is among the finest in the world. The bureau publishes also excess mortality. They are below average at the moment also for 65 and older. As a matter of coincidence, the New York times from today has just put a graph of the excess death rates from 2018-2021 on the front page.



Population growth

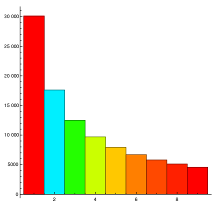
34.9. The simplest model for population dynamics is **exponential growth model** given by the $f(t) = e^{ct}$, where c is a constant. In epidemiology, it is custom to define $f(t+h)/f(t) = R_0$, where R_0 is the **reproduction number** and h is an **infect period time**. This means with the notation introduced in the first week that $Df(t) = f(t+h) - f(t) = (R_0 - 1)f(t)$ so that $f(t) = (1+h)^{(R_0-1)t} f(0) = e^{ct} f(0)$ with $c = (R_0 - 1)\log(1+h)$. In the news, we often see R_0 values but not the **mean infection period** which is also important for estimating the growth.

34.10. The function $f(t) = e^{ct}$ satisfies the differential equation $f'(t) = cf(t)$. We are not going more into differential equation but just want to point out that models usually only work in specific situations. Exponential growth models for example fail always after some time, simply because resources are finite. A population of rabbits will grow exponentially fast at first, but then, after some saturation has been reached, the exponential model (famously illustrated by the Fibonacci rabbits) can not be sustained and a better model will be needed.

34.11. The text book model is the **logistic growth** like $f'(t) = f(t)(1 - f(t))$ which is a **differential equation**. You can check that the function $f(t) = \frac{e^t}{e^t+1}$ solves this equation. Can you do that? You have to check that the derivative $f'(t)$ agrees with $f(t)(1 - f(t))$. The logistic model is much better. Do you remember discrete version, the **logistic equation**. It turns out that differential equations $x'(t) = F(x(t))$ in one dimension can not exhibit complicated behavior but that maps $x(t+1) = F(x(t))$ can do. The case $x(t+1) = 4x(t)(1-x(x))$ has illustrated this.

Benford's law

34.12. If you look at random data and make a statistics of the **first significant digit**, there is a surprise, as Simon Newcomb and Frank Benford realized in 1881 or 1938 respectively. The digit 1 appears more frequent than the number 2. One can look at stock market values, lengths of rivers or car prizes, one sees similar data. Here is an example, where one can prove the distribution to be true [2]



Homework

Problem 33.1: In the movie **Fermat's room**, a string of 169 zeros and one's are given. You are a mathematician in that movie and need to figure out what the numbers mean, otherwise, the room gets crushed. Can you figure it out? $\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 1,$
$1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 1, 1, 0,$
$0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 0, 0, 1, 1, 1, 1,$
$0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0,$
$1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1,$
$1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$.

Problem 33.2: Verify that the function $\frac{e^{4t}}{e^{4t}+1}$ satisfies the logistic equation $f'(t) = 4f(t)(1-f(t))$.

Problem 33.3: Make a list of the first 30 Fibonacci numbers and make a statistics about the first significant digit.

Problem 33.4: Look up the population of the numbers of at least 20 largest cities on Wikipedia.

Problem 33.5: Now make a statistics about the list the first significant digits with the data you have collected in 33.4.

Oliver Knill, knill@math.harvard.edu, Math 1a, Harvard College, Spring 2021

[2] This example appears in V. Arnold's, Mathematical methods of Mechanics

INTRODUCTION TO CALCULUS

MATH 1A

Unit 35: Review

The older review sheets are contained in unit 15 and unit 28. We do here not recopy those parts.

Substitution

Substitution replaces $\int f(x)\,dx$ with $\int g(u)\,du$ with $u = u(x), du = u'(x)dx$. Cases:

A) The integral of $f(x) = g(u(x))u'(x)$, is $G(u(x))$ where G is the anti derivative of g.
B) $\int f(ax+b)\,dx = F(ax+b)/a$ where F is the anti derivative of f.

Examples:
A) $\int \sin(x^5)x^4\,dx = \int \sin(u)\,du/5 = -\cos(u)/5 + C = -\cos(x^5)/5 + C$.
B) $\int \log(5x+7)\,dx = \int \log(u)\,du/5 = (u\log(u) - u)/5 + C = (5x+7)\log(5x+7) - (5x+7) + C$.

Integration by parts

A) Direct (use this if it can be done in one step)

$$\int \text{x} \sin(x)\,dx = \text{x}\,(-\cos(x)) - \int 1\,(-\cos(x))\,dx = -x\cos(x) + \sin(x) + C\,dx\,.$$

B) Tic-Tac-Toe: to integrate $x^2\sin(x)$:

x^2	$\sin(x)$	
$2x$	$-\cos(x)$	$\oplus$
2	$-\sin(x)$	$\ominus$
0	$\cos(x)$	$\oplus$

The anti-derivative is

$$-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C\,.$$

C) Merry go round: Example $I = \int \sin(x)e^x\,dx$. Use parts twice and solve for I.

Partial fractions

A) Make a common denominator on the right hand side $\frac{1}{(x-a)(x-b)} = \frac{A(x-b)+B(x-a)}{(x-a)(x-b)}$. and compare coefficients $1 = Ax - Ab + Bx - Ba$ to get $A + B = 0, Ab - Ba = 1$ and solve for A, B.
B) The residue method works better if $f(x) = p(x)/(x-a)(x-b)$ with different a, b, the coefficients A, B in $\frac{p(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$ can be obtained from

$$A = \lim_{x \to a}(x-a)f(x) = p(a)/(a-b),\ B = \lim_{x \to b}(x-b)f(x) = p(b)/(b-a) .$$

Examples:
A) $\int \frac{1}{(x+1)(x+2)}\, dx = \int \frac{A}{x+1}\, dx + \int \frac{B}{x+2}\, dx$. Find A, B by multiplying out and comparing coefficients in the nominator.
B) Directly write down $A = 1$ and $B = -1$, by plugging in $x = -2$ after multiplying with $x - 2$. or plugging in $x = -1$ after multiplying with $x - 1$.

Improper integrals

A) Integrate over infinite domain.
B) Integrate over point where f is unbounded.

Examples:
A) $\int_0^\infty 1/(1+x^2) = \arctan(\infty) - \arctan(0) = \pi/2 - 0 = \pi/2$.
B) $\int_0^1 1/x^{2/3}\, dx = (3/1)x^{1/3}|_0^1 = 3$.

Trig substitution

When integrating function like $\sqrt{1-x^2}$, replace x by $\sin(u)$. **Example:**
$\int_{-1}^1 \sqrt{1-x^2}\, dx = \int_{-\pi/2}^{\pi/2} \cos(u)\cos(u) du = \int_{-\pi/2}^{\pi/2} (1+\cos(2u))/2 = \frac{\pi}{2}$.

Terminology in Application

Music: hull function, piano function
Economics: average cost, marginal cost and total cost. Strawberry theorem
Operations research: find extrema, critical points, derivative tests
Computer science: random function from given function
Statistics: PDF, cumulative distribution function, expectation, variance.
Distributions: normal distribution, geometric distribution, Cauchy distribution.
Geometry: area between two curves, volume of solid
Numerical integration: Riemann sum, trapezoid rule, Simpson rule, Monte Carlo
Root finding: Bisection method, Newton method $T(x) = x - f(x)/f'(x)$.
Psychology: critical points and catastrophes.

Physics: position, velocity and acceleration, work and power.
Gastronomy: turn table to prevent wobbling, bottle calibration.

Checklists:

Integral techniques to consider

Try in the following order:

- [ ] Know the integral
- [ ] Substitution
- [ ] Trig substitution
- [ ] Integration by parts
- [ ] Partial fractions

Especially:

- [ ] Tic-Tac-Toe for integration by parts
- [ ] Hospital Method for partial fractions
- [ ] Merry go round method for parts

Integrals to know well

- [ ] $\sin(x)$
- [ ] $\cos(x)$
- [ ] $\tan(x)$
- [ ] $\log(x)$
- [ ] $\exp(x)$
- [ ] $1/x$
- [ ] $1/x^n$
- [ ] x^n
- [ ] $\sqrt{x}$
- [ ] $1/\cos^2(x)$
- [ ] $1/\sin^2(x)$
- [ ] $1/(1+x^2)$
- [ ] $1/(1-x^2)$
- [ ] $1/\sqrt{1-x^2}$
- [ ] $\sqrt{1-x^2}$

Applications you have to know

- **Derivative:** Limit of differences $D_h f = [f(x+h) - f(x)]/h$ for $h \to 0$
- **Integral:** Limit of Riemann sums $S_h f = [f(0) + f(h) + \cdots + f(kh)]h$.
- **Newton step:** $T(x) = x - f(x)/f'(x)$.
- **Marginal cost:** the derivative F' of the total cost F.
- **Average cost:** F/x where F is the total cost.
- **Velocity:** Derivative of the position.
- **Acceleration:** Derivative of the velocity.
- **Curvature:** $f''(x)/(1+f'(x)^2)^{3/2}$.
- **Probability distribution function:** non-negative function with total $\int f(x)dx = 1$.
- **Cumulative distribution function:** anti-derivative of the PDF.
- **Expectation:** $\int xf(x)\, dx$, where f is the probability density function.
- **Piano function:** frequencies $f(k) = 440 \cdot 2^{k/12}$ for integer k.
- **Hull function:** $\sin(x)\sin(10000x)$ has hull $|\sin(x)|$
- **Catastrophe:** A parameter c at which a local minimum disappears.

Core concepts

- **Fundamental:** The fundamental theorem of calculus
- **Extrema:** Second derivative test
- **Derivatives:** slope rate of change
- **Integrals:** area, volume
- **Limits:** Hospital!
- **Continuity:** know the enemies of continuity
- **Numerics:** Riemann sum, Trapezoid and Simpson rule
- **Rules:** Differentiation and integration rules.
- **Methods:** Integration by parts, Substitution, Partial fraction.

Not needed on your fingertips but fair game

- **Epidemic:** logistic growth.
- **Entropy:** $-\int f(x)\log(f(x))\, dx$.
- **Monte Carlo integration:** $S_n = \frac{1}{n}\sum_{k=1}^{n} f(x_k)$, where x_k are random in $[a,b]$.
- **Bart Simpson rule:** $S_n = \frac{1}{6n}\sum_{k=1}^{n}[f(x_k) + 4f(y_k) + f(x_{k+1})]$.
- **Cocktail party stuff:** Eat,integrate and love.
- **Bottles:** How to calibrate bottles. The calibration formula.
- **Sofia:** The name of a calculus bot once living in the math department.
- **Wobbly chair:** One can turn a chair on any lawn to stop it from wobbling.
- **Song:** The hit: "low d high take high d low, cross the line and square the low"
- **1a Song:** The golden record: "Good bye Math 1A, this is the end of the season"
- **Midi function:** $f(m) = 440 * 2^{(m-69)/12}$.

Lecture 36: Related rates (not covered in 2020/21)

36.1. Here is a topic on differentiation. It allows us to review the **chain rule**

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

a rule which will lead to the "substitution" integration technique. Related rates are often perceived difficult. The reason is that many of these problems are **word problems** which sometimes are formulated in an ambiguous or misleading way. This is not so much the problem of textbooks, it is an inherent problem to deal with situations, where functions are also variables.

36.2. If a variable depends on an other variable (like a radius of a deflating balloon depends on time), then one has to be very clear what we mean with notions like "variable" or "function". We have seen in the chain rule the Smeagol-Golum type transition. The good news is that the topic of related rates is actually part of an other universe, the calculus of **several variable**. It actually belongs there. This misplacement is an other reason why "everybody hates, related rates". By the way, if you learn multi-variable calculus, then these problems are very clear and easy.

36.3. The topic of **related rates** deals with relations of various variables. Since it is actually a multi-variable topic, there is no harm done by skipping the topic at first. It is well known that everybody hates related rates. Seriously, the problem is not so much the set-up. The problem is that the topic often comes with incomprehensible word problems. It is a matter of fact that writing clear related rates problems is hard. We use here only well tested classic problems and try to make super clear what is a variable and what is a function.

36.4. Differentiation gives a relation between the derivatives (rate of change). In all these problems, we have an **equation** and a **rate**. You can then solve for the rate which is asked for. We make the problems here a bit easier by telling explicitly what the equation and what the rate is and what we are looking for.

Example: Hydrophilic **water gel spheres** have volume $\boxed{V(r(t)) = 4\pi r(t)^3/3}$ and expand at a rate $\boxed{V' = 30}$. Find $r'(t)$. **Solution:** $30 = 4\pi r^2 r'$. We get $r' = 30/(4\pi r^2)$.

Example: A **wine glass** has a shape $y = x^2$ and volume $\boxed{V(y) = y^2\pi/2}$. Assume we slurp the wine with constant rate $\boxed{V' = -0.1}$. With which speed does the height decrease? We have $d/dtV(y(t)) = V'(y)y'(t) = \pi y y'(t)$ so that $y'(t) = -1/(\pi y)$.

Example: A **ladder** has length 1. Assume it slips to the ground away with constant speed $\boxed{x' = 2}$. What is the speed of the top part of the ladder which slides down the wall at the time when $x = y$ if $\boxed{x^2(t) + y^2(t) = 1}$?
Solution: differentiation gives $2x(t)x'(t)+2y(t)y'(t) = 0$. We get $y'(t) = -x'(t)x(t)/y(t) = 2 \cdot 1 = 1$.
Example: A **kid slides** down a slide of the shape $\boxed{y = 2/x}$. Assume $\boxed{y' = -7}$. What is $x'(t)$? Evaluate it at $x = 1$.
Solution: differentiate the relation to get $y' = -2x'/x^2$. Now solve for x' to get $x' = -y'x^2/2 = 7/2$.
Example: A **canister of oil** releases oil so that the area grows at a constant rate $\boxed{A' = 5}$. With what rate does the radius increase?
Solution. We have $\boxed{A(r) = r^2\pi}$ and so $5 = A'(r) = 2rr'\pi$. Solving for r' gives $r' = 5/(2r\pi)$.

Related rates problems link quantities by a $\boxed{\text{rule}}$. These quantities can depend on time. To solve a related rates problem, differentiate the $\boxed{\text{rule}}$ with respect to time use the given $\boxed{\text{rate of change}}$ and solve for the unknown rate of change. Since related change problems are often difficult to parse. We have the $\boxed{\text{rule}}$ and given $\boxed{\text{rate of change}}$ boxed.

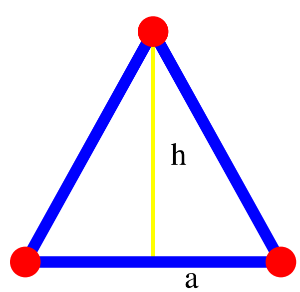
Homework

Problem 36.1: The **ideal gas law** $\boxed{pV = T}$ relates pressure p and volume V and temperature T. Assume the temperature $T = 50$ is fixed and $\boxed{V' = -5}$. Find the rate p' with which the pressure increases if $V = 10$ and $p = 5$.

Problem 36.2: Assume the **total production rate** P of an Oculus Rift is constant $P = 100$ and given by the **Cobb-Douglas formula** $\boxed{P = L^{1/3}K^{2/3}}$. Assume labor is increased at a rate $\boxed{L' = 2}$. What is the cost change K'? Evaluate this at $K = 125$ and $L = 64$.

Problem 36.3: You observe an **airplane** at height $h = 10'000$ meters directly above you and see that it moves with rate $\boxed{\phi' = 5\pi/180 = \pi/36}$ radiants per second (which means 5 degrees per second). What is the speed x' of the airplane directly above you where $x = 0$? Hint: Use $\boxed{\tan(\phi) = x/h}$ to get ϕ for $x = 0$.

Problem 36.4: An **isosceles triangle** with base $2a$ and height h has fixed area $\boxed{A = ah = 1}$. Assume the height is decreased by a rate $\boxed{h' = -2}$. With what rate does a increase if $h = 1/2$?



Problem 36.5: There are **cosmological models** which see our universe as a four dimensional sphere which expands in space time. Assume the volume $\boxed{V = \pi^2 r^4/2}$ increases at a rate $\boxed{V' = 100\pi^2 r^2}$. What is r'? Evaluate it for $r = 47$ (billion light years).

INTRODUCTION TO CALCULUS

MATH 1A

Data project 1: Primes

1.1. Define the **prime function** $f(x)$ as the function on positive integers which gives the x'th prime. So,

$$f(1) = 2, f(2) = 3, f(3) = 5, f(4) = 7 \; .$$

By definition, the prime function is **monotone** in the sense that $f(x+1) > f(x)$ and more generally $f(y) > f(x)$ if $y > x$. To start this project, make a list of the first 17 primes.

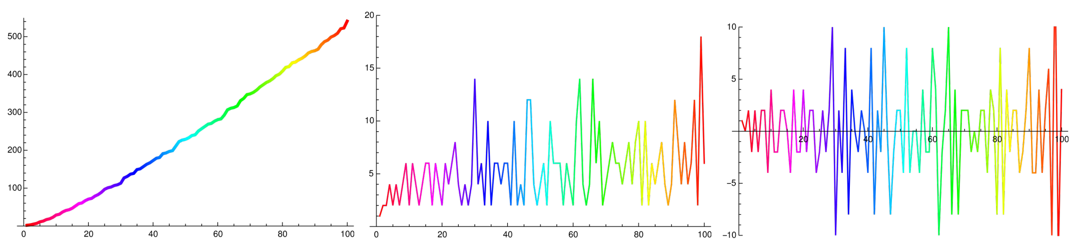


FIGURE 1. Prime function $f(x)$, velocity $f'(x) = f(x+1) - f(x)$ and acceleration $f''(x) = f(x+2) - 2f(x+1) - f(x)$.

1.2. Define the **derivative**

$$f'(x) = f(x+1) - f(x) \; .$$

This measures the increase or decrease between x and $x+1$ and is also known as **rate of change**. For any positive integer h, the expression

$$\frac{f(x+h) - f(x)}{h}$$

is called the **average rate of change** from x to $x+h$. The name average is justified because

$$\frac{f(x+h) - f(x)}{h} = \frac{f'(x) + f'(x+1) + \cdots + f'(x+h-1)}{h}$$

is the average between all the rate of changes. Question 2:
write down the average rate of change equation for $x = 1$ and $h = 9$.

1.3. A point x for which $f'(x) = 2$ is called a **prime twin**. A big conjecture is that there are infinitely many prime twins. The problem appears too difficult for current mathematics to be solved; but we can find small prime twins, especially with a computer. Third question: find the first 10 prime twins

1.4. The second derivative of f is called the **acceleration**. It is defined as the derivative of the derivative function f'. We have

$$f''(x) = f'(x+1) - f'(x) = (f(x+2) - f(x)) - (f(x+1) - f(x) = f(x+2) - 2f(x+1) + f(x) .$$

We have $f''(2) = 0$ because $f(4) - 2f(3) + f(2) = 7 - 2 * 5 + 3 = 0$. Fourth problem: $\boxed{\text{find the next two } x \text{ for which } f''(x) = 0}$.

1.5. The **fundamental theorem of calculus for data** tells that if $f'(x) = f(x + 1) - f(x)$, then

$$f'(a) + f'(a + 1) + f'(a + 2) + \cdots + f'(b - 1) = f(b) - f(a)$$

for any integers a, b. $\boxed{\text{Verify this for the prime function } f(x) \text{ with } a = 1, b = 10}$. The fundamental theorem will later in the course be written as

$$\int_a^b f'(x)dx = f(b) - f(a) .$$

1.6. Primes are one of the first things which have been considered in mathematics. Euclid was the first to prove that there are infinitely many primes. His argument was ingenious: assume there would be only finitely many prime data $p_1, \dots, p_n$, then the number $m = p_1 \cdot p_2 \cdots p_n + 1$ has either to be prime or to contain a new prime factor different from the given list; indeed, m is not divisible by any of the primes. This is result is important as it justifies that the **prime function** f we have been talking about, really exists. In the last part, we want you to look up **Sophie Germain primes**. Fifth problem: $\boxed{\text{Look up the definition Germain primes.}}$ $\boxed{\text{Why was Sophie interested in these primes?}}$

1.7. Finally, in problem 6, we want you to look up the largest known prime. There is a large collaborative project going on in which humanity tries to push the largest known prime further. $\boxed{\text{What is the largest known prime today?}}$ $\boxed{\text{What is the name of this prime hunting project?}}$

FIGURE 2. An artistic rendering of Sophie Germain.

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Data project 2: Polyhedra

2.1. In the context of **data science**, **graphs** are very important. **Graph databases** started to appear already with the **Incas** in the form of **Khipu** documents. Nature has used DNA as a database since millions of years. In modern information technology, graph databases replace more and more traditional relational databases. A **graph** is a collection of nodes called **vertices** joined by connections called **edges**. Street maps, electric grids, directories, family or collaboration trees, brains, social networks or polyhedra are examples of graphs.

2.2. A graph G comes with a **graph polynomial**. This brings us to calculus. If the graph has a_0 vertices and a_1 edges and a_2 triangles the polynomial is defined as $f_G(x) = 1 + a_0x + a_1x^2 + a_2x^3$. More generally, if a_k is the number of k-**cliques**, groups of k vertices all connected to each other, then $f_G(x) = 1 + \sum_{k=0} a_k x^{k+1}$, where Σ is the summation sign.

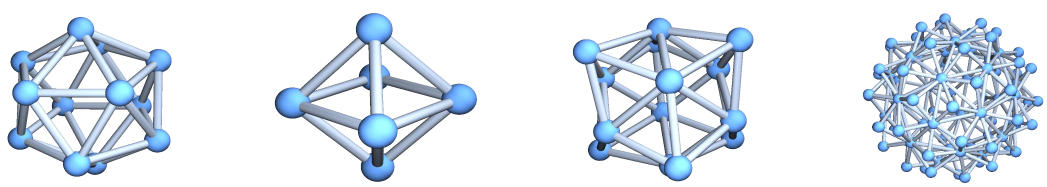


FIGURE 1. The Icosahedron, Octahedron, Tetraxishexahedron and Echidnahedron.

2.3. The **unit sphere** $S(v)$ of a vertex v consists of the subgraph build by all vertices connected to the node. If G is a friendship graph, and v is you, then $S(v)$ is the friend network of your friends without you. Let $f_{S(v)}(x)$ denote the polynomial function to $S(v)$.

Theorem: $f'_G(x) = \sum_{v \in V} f_{S(v)}(x)$

It is a **Gauss-Bonnet theorem** [1] because it allows to compute the Euler characteristic $\chi(G) = a_0 - a_1 + a_2 - \cdots$ of a graph by integrating and evaluating at -1: Indeed, we have from the **fundamental theorem of calculus** $\chi(G) = f_G(0) - f_G(-1) =$

[1] O. Knill, Dehn-Sommerville from Gauss-Bonnet, https://arxiv.org/abs/1905.04831, 2019

$\int_{-1}^0 f'_G(x)\, dx$. Because the formula computes $f_G(x)$ recursively through unit spheres, allows to compute clique numbers of most graphs quickly. In general, it is a notoriously difficult NP complete problem.

2.4. Example: the icosahedron G has 12 vertices, 30 edges and 20 triangles. The polynomial is $f_G(x) = 1 + 12x + 30x^2 + 20x^3$. And the Euler characteristic is $\chi(G) = f(0) - f(-1) = 12 - 30 + 20 = 2$. René Descartes already noticed that Platonic solids lead to $\chi(G) = 2$ and wrote it down in an encrypted way in a secret notebook since mathematical theorems were at that time considered treasures. The unit sphere of a vertex is a circular graph with 5 vertices and 5 edges has the polynomial $f_{S(v)}(x) = 1 + 5x + 5x^2$. Summing over all 12 vertices gives $12 + 60x + 60x^2$. This agrees with $f'_G(x) = 12 + 60x + 60x^2$. Integrate this from -1 to 0 gives the already computed $\chi(G) = 2$.

2.5. Example: for the Echidnahedron with 92 vertices, 270 edges, 240 triangles and 60 tetrahedra, the polynomial is $f_G(x) = 1 + 92x + 270x^2 + 240x^3 + 60x^4$. The unit spheres are 60 triangles with $f_{S(v)}(x) = 1 + 3x + 3x^2 + x^3$ then 12 unit spheres with $f_{S(v)}(x) = 1+10x+15x^2+5x^3$ and 20 unit spheres with $f_{S(v)}(x) = 1+12x+18x^2+6x^3$. They all sum up to $92 + 540x + 720x^2 + 240x^3$ which agrees with $f'_G(x)$.

2.6. Problem 1: Repeat the computation for the octahedron G. So, compute $f_G(x)$, find the unit spheres $S(v)$ and compute $f_{S(v)}(x)$ the verify the formula from the theorem and compute the Euler characteristic.

2.7. Problem 2: Do the same for the **tetraxis hexahedron**: find $f_G(x)$ and $\chi(G)$. It is a cube on which we add a central node on each face and connect it to the 4 vertices of the square. So there are $8 + 6 = 14$ vertices and $6 * 4$ triangles.

2.8. The graphs of the polynomials are very interesting. One can study their roots, their maxima minima etc. When plotting the functions $f_G(x)$ for polynomials, we often notice some symmetry. This is called a **Dehn-Sommerville** property. Let us call a graph a **polyhedron** if all unit spheres $S(v)$ are circular graphs of 4 or more vertices.

2.9. Problem 3: Plot the graphs of $f_G(x)$ for the polyhedra: icosahedron, octahedron and tetraxis hexahedron.

Corollary: If G is a polyhedron, then $f_G(x)$ has a root at $x = -1/2$.

This follows from the theorem. All $g(x) = f_{S(v)}(x)$ have the property that $g(x + 1/2)$ even. Therefore the sum of such functions is even and so $f'_G(x+1/2)$ is even. But that means that $f_G(x + 1/2)$ is odd, so that $x = -1/2$ is a root.

2.10. If G is a d-dimensional Dehn-Sommerville space define the function $h_G(x) = (x - 1)^{d+1} f_G(1/(x - 1))$. In our case, if triangles are the largest cliques, then $d = 2$ and $h_G(x) = (x - 1)^3 f_G(1/(x - 1))$. The **Dehn-Sommerville relations** assert that the h-vector is **palindromic**, meaning that $h_i = h_{d+1-i}$ for all $i = 0, \dots, d + 1$. These identities is what the Dehn and Sommerville have noticed first. If G satisfies Dehn-Sommerville, then $f(-1) = (-1)^d f(0) = (-1)^d$ and so $\chi(G) = 1 - f(-1) = 1 + (-1)^d$. In the case $d = 2$, we get $\chi(G) = 2$. For the icosahedron, $f_G(x) = 1+12x+30x^2+20x^3$, we have $h_G(x) = 1 + 9x + 9x^2 + x^3$ and indeed $(1, 9, 9, 1)$ is a polindrome. **Problem 4:** Compute $h_G(x)$ for the octahedron.

2.11. The theorem we have seen is very general and holds for any network. **Problem 5:** Draw your own graph (chose something you can actually do), then build all functions $f_{S(v)}(x)$ and compare their sum with the derivative $f'_G(x)$.

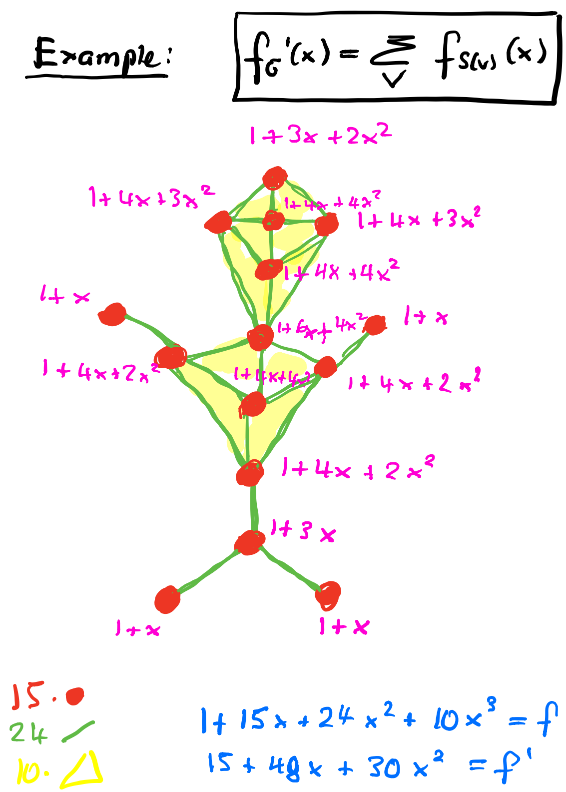


FIGURE 2. A hand-drawn graph. We compute at every vertex v the function $f_{S(v)}(x)$ and them up to get the derivative of $f_G(x)$. Pretty cool. The coefficients of the linear part produce the **Euler Handshake theorem** telling that 2 times the number of edges is the sum over all vertex degrees.

OLIVER KNILL, KNILL@MATH.HARVARD.EDU, MATH 1A, HARVARD COLLEGE, SPRING 2021

Data project 3: Chaos

3.1. Given a function, like $f(x) = 4x(1-x)$, we can look for points where $f(x) = x$. This is called a **fixed point**. You can see that in our case, $x = 0$ and $x = 3/4$ are the two solutions. The function $f(f(x))$ is a new function and in our example again a polynomial. Now, if $f(f(x)) = x$, then we have a periodic point of period 2 because if we apply f to such a point, we return after 2 steps, $x, f(x), f(f(x)) = x$.

3.2. Problem 1: when expanding $f(f(x))$ we get a polynomial of degree 4. $f(f(x)) = 4(4x(1-x))(1-(4x(1-x))) = 16x - 80x^2 + 128x^3 - 64x^4$. Find all the roots of this function. You can use that you already know two of them since if $f(x) = x$, then also $f(f(x)) = x$.

3.3. Let us start with $x = 0.4$ and apply $f(x) = 4x(1-x)$. We get 0.96. Now compute $f(0.96) = 1.536$. We do that 60 times. Here is the Mathematica code which does that.

```
f[x_] := 4x(1-x); NestList[f, 0.4, 60]
```

Running this code produces the following data 0.4, 0.96, 0.1536, 0.520028, 0.998395, 0.00640774, 0.0254667, 0.0992726, 0.35767, 0.918969, 0.29786, 0.836557, 0.546917, 0 0.991195, 0.034909, 0.134761, 0.466403, 0.995485, 0.0179785, 0.0706211, 0.262535, 0.774441, 0.698727, 0.84203, 0.532063, 0.995888, 0.0163812, 0.0644512, 0.241189, 0.732068, 0.784578, 0.676061, 0.87601, 0.434465, 0.982821, 0.067536, 0.251899, 0.753784, 0.742374, 0.76502, 0.719059, 0.808053, 0.620412, 0.942003, 0.218532, 0.683103, 0.865893, 0.464488, 0.994956, 0.0200758, 0.0786912, 0.289996, 0.823593, 0.581151, 0.973658, 0.102592, 0.368269, 0.930587, 0.258378, 0.766475, 0.715963.

3.4. We start again with $x = 0.4$ again but apply the function $g(x) = 4x - 4x^2$. We get 0.96. Now compute $f(0.96) = 1.536$ etc. We do that 60 times. Here is the Mathematica code which does that.

```
g[x_] := 4x-4x^2; NestList[g, 0.4, 60]
```

Running this line of code produces the following data 0.4, 0.96, 0.1536, 0.520028, 0.998395, 0.00640774, 0.0254667, 0.0992726, 0.35767, 0.918969, 0.29786, 0.836557, 0.546917, 0.991195, 0.034909, 0.134761, 0.466403, 0.995485, 0.0179785, 0.0706211, 0.262535, 0.774441, 0.698727, 0.84203, 0.532063, 0.995888, 0.0163812, 0.0644512, 0.241189, 0.732068, 0.784578, 0.676061, 0.87601, 0.434465, 0.982821, 0.0675375, 0.251905, 0.753795, 0.742353, 0.76506, 0.71 8973, 0.808203, 0.620042, 0.942359, 0.217273, 0.680263, 0.870022, 0.452336, 0.990913, 0.0360196, 0.138889, 0.478395, 0.998133, 0.00745456, 0.029596, 0.11488, 0.406731, 0.965204, 0.134342, 0.465178, 0.99515.

3.5. Problem 2) a) What is algebraically the difference between $f(x)$ and $g(x)$? Are you surprised when you look at the data? We have prepared a Desmos applet

https://www.desmos.com/calculator/xvh5ae0a1a

For you to explore that. b) What does Desmos give for $f^{(60)}(0.4)$? c) What result does Desmos give for $g^{(60)}(0.4)$? Note that we write $f^{(2)}(x) = f(f(x))$ and $f^{(3)}(x) = f(f(f(x)))$ etc.

3.6. As the title of this project shows, the reason for this strange behavior is "chaos". Chaos is defined as "sensitive dependence on initial conditions".

3.7. We now take a graphing calculator or the scientific calculator on your phone to experiment with iterating functions.

Problem 3) a) What happens if you push the $\boxed{\text{cos}}$ button repetitively. b) What happens if you push alternating the $\boxed{\text{tan}}$ and $1/x$ button? (This actually means that we are iterating the cot function but there is no cot button in general on calculators).
Problem 4) a) Who was Mitchell Feigenbaum? b) What did he discover? c) Where is his discovery applied?

Problem 5) a) What is the **Feigenbaum bifurcation diagram** for the logistic map $f_c(x) = cx(1-x)$ describe? b) What is the **Feigenbaum constant?**

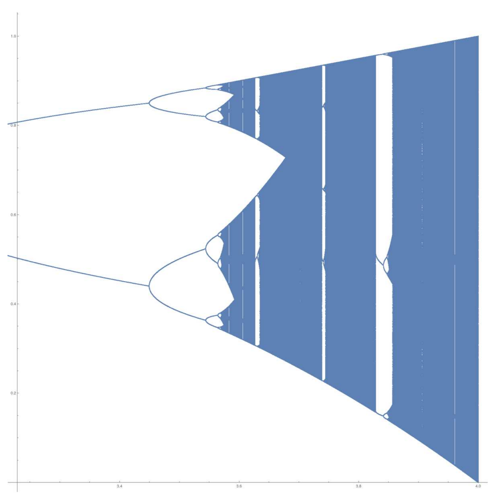


FIGURE 1. Feigenbaum bifurcation diagram

OLIVER KNILL, KNILL@MATH.HARVARD.EDU, MATH 1A, HARVARD COLLEGE, SPRING 2021

Data project 4: Monte Carlo

4.1. Instead of the Riemann integral $S_n = \sum_{a \leq \frac{k}{n} < b} f(\frac{k}{n})\frac{1}{n}$ approximating the integral $\int_a^b f(x)\, dx$, we can also chose random numbers x_k and produce $S_n = \sum_{k=1}^n f(x_k)\frac{b-a}{n}$. Computing the integral as such is called a **Monte Carlo** integration. It turns out to be equivalent to the **Lebesgue integral**, a more sophisticated integral which is required in real analysis or probability theory.

4.2. How do we generate random numbers? There are two fundamental types. The first uses a **dynamical system** to generate pseudo random numbers. An example is to start with a **seed** $x_0 = 0.4$ (which could also depend on the time or other metrics in the computer) and then compute $g^j(x)$ where $g(x)$ is a chaotic map like $g(x) = 4x(1-x)$. The random number generator can then be seeded like $k = 234$. The random numbers x_j are then $x_j = g^{k+j}$ where j goes from 1 to n. This is not bad as we have seen in the last project that it even matters whether we take $g(x) = 4x(1-x)$ or $g(x) = 4x - 4x^2$. Here is code:

```
x0=0.4; k=234; n=10; g[x_]:=4x(1-x);
x1=Last[NestList[g,x0,k-1]]; NestList[g,x1,n-1]
```

4.3. Pseudo random numbers could also be generated (a bit more costly of course) from the digits of pi.

```
n=1000; X[k_]:=Mod[N[Pi*10^k,2n],1];
f[x_]:=Sin[x]; a=0; b=Pi; Sum[f[Pi*X[k]],{k,1,n}]*(b-a)/n
```

Problem 1. Use at least 20 digits of π to find a numerical approximation of the integral $\int_0^1 x^2\, dx$. Make groups of 2 digits to form numbers. Start with the digits after the decimal point. The first number $x_1 = 14/100$, the second $x_2 = 15/100$. Now write down the Riemann sum and give the result.

4.4. Problem 2. We want you to compute the area of the Mandelbrot set. You can do that by hand, printing out a version in which the coordinate axes are given and splitting it up into a grid, then counting. You can also use a machine. Here is some Mathematica code:

```
M=Compile[{x,y},Module[{z=x+I y,k=0}, While[Less[Abs[z],2]
  &&Less[k,1000],z=N[z^2+x+I y];++k]; Floor[k/1000]]];
9*Sum[M[-2+3 Random[],-1.5+3 Random[]],{100000}]/100000
```

On an Online page like *https : //www.tutorialspoint.com/execute_python_online.php* you can also execute some Python code. If there is somebody in your group who knows how to run Python locally, you might get better results. The code should be pretty self-explanatory. If you are interested 1j denotes i. We shoot random points into a square of side length 3 (and area 9) and count the number of hits of the Mandelbrot set.

```
from numpy import random
import math

iterations = 200
samples = 1000

def M(x,y):
    z = x+y*1j
    k = 0
    while abs(z)<2 and k<iterations:
      z = z*z+ x+y*1j
      k += 1
    return math.floor(k/iterations)

sum = 0
n = 0
while n < samples:
   n += 1
   sum += M(-2 + 3*random.rand(), -1.5 +3*random.rand())

final = 9*sum/samples
print(final);
```

The Mandelbrot set can be generated quickest with the open source ray tracer Povray:

```
camera{location <-1,0,3> look_at <-1,0,-3> right <0,16/9,0>
up <0,0,1>} plane{z,0 pigment{mandel 200 color_map{[0 rgb 0]
[1/6 rgb <1,0,1>][1/3 rgb <1,0,0>][1 rgb 0]}} finish{ambient 1}}
```

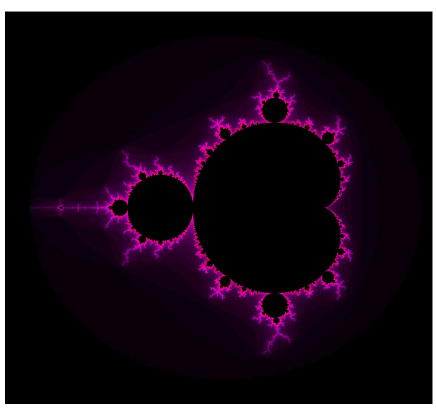


FIGURE 1. The Mandelbrot set.

4.5. Task: solve the two short problems. Author a 2 slide presentation or a document with two pages in which the two problems are solved, then submit this as a PDF. Please include the names of each of the group members who worked on the project.

Oliver Knill, knill@math.harvard.edu, Math 1a, Harvard College, Spring 2021

3/5/2021: First hourly

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as a PDF, handwritten in one file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids or online tools or external information are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 6 AM on March 6th. Do not communicate with anybody in the class during the exam period.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F The function $f(x) = \log(\exp(x))$ has a root at $x = 0$.

2) T F The "today" function $f(x) = x^3 - x^5 + x^{2021}$ has a real root.

3) T F $\cot(\pi/4) = 1$.

4) T F $f(x) = |2 + \sin(x)|$ is differentiable everywhere.

5) T F The chain rule assures that $\frac{d}{dx} \sin(\sin(x)) = \cos(\cos(x)) \cos(x)$.

6) T F $f(x) = 1 + \frac{\sin(x)^2}{x}$ with assumption $f(0) = 1$ is continuous.

7) T F A differentiable function satisfying $f(0) = 1, f(1) = 0$ must have $f'(x)$ be negative somewhere in $[0, 1]$.

8) T F $\sec(x)$ is the inverse function of $\cos(x)$.

9) T F If $f(x)$ is differentiable at 0, and $f(0) = 0$, then $f(f(x))$ is differentiable at 0.

10) T F $f(x) = x^5/(1 + x^5)$ defines an indeterminate form at $x = 0$.

11) T F $f(x) = (\sin(\pi/2+h)-\sin(\pi/2))/h < 0$ for all positive h smaller than $\pi/2$.

12) T F The function $f(x) = x^7$ has a critical point at $x = 0$.

13) T F If f has an inflection point at 0, then $g(x) = 1 - f(x)$ has an inflection point at 0.

14) T F $f(x) = \cot(x^2)$ has a vertical asymptote at $x = 0$.

15) T F $f(x) = \sin(x)$ takes the value $\pi/2$ at some point x.

16) T F If $f(x) = x(x - 1)$, then $Df(x) = f(x + 1) - f(x) = 2x$.

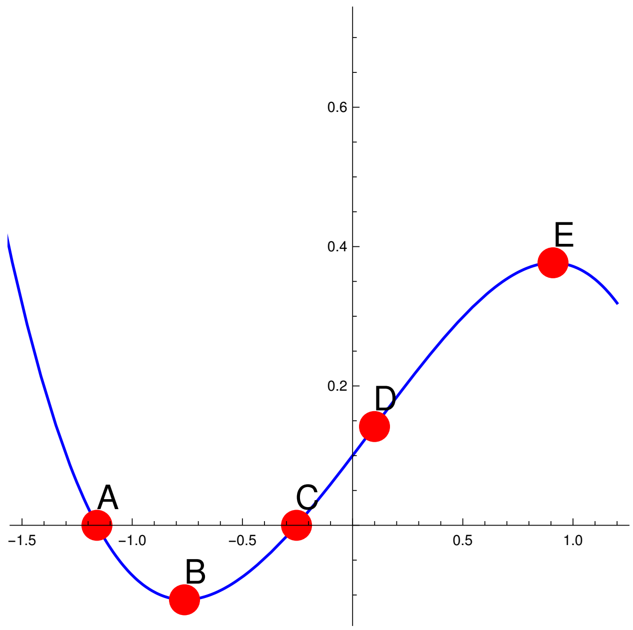
17) T F The function $f(x) = \sin(x) \sin(1/x)$ with the understanding $f(0) = 0$ is continuous everywhere.

18) T F $\frac{d}{dx} \arctan(x^2) = 2x/(1 + x^2)$.

19) T F $\frac{d}{dx} \log(7 - x) = -1/(7 - x)$.

20) T F The derivative of g/f is $(fg' - f'g)/f^2$.

Problem 2) Choice problem (10 points) No justifications are needed.



a) (2 points) List the points A-E which are roots of f.

b) (2 points) List the points A-E that are critical points of f.

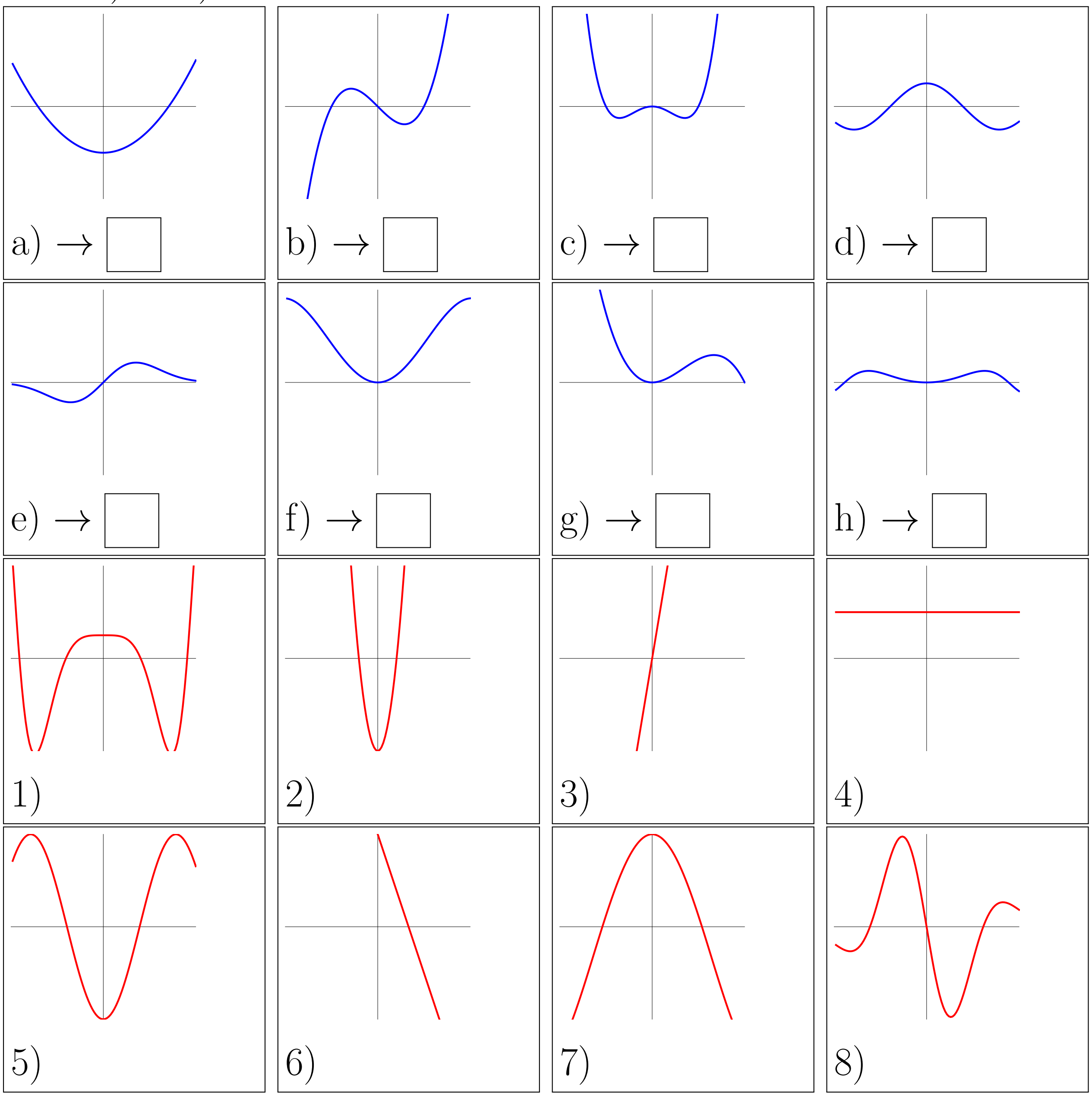
c) (2 points) List the points A-E that are critical points of f'.

d) (2 points) List the critical points A-E that are local maxima.

e) (2 points) List the points A-E of f that are global minima on the given interval.

Problem 3) Matching problem (10 points) No justifications are needed.

Match the functions f a) to h) with the second derivatives functions 1) to 8).



Problem 4) Continuity (10 points)

Which of the following functions are continuous on $[-1, 1]$? As always, we extend continuity to functions for which a continuation is possible to initially not defined points, like $f(x) = (x^2-1)/(x-1)$, which was considered continuous because with $f(1) = 2$, it becomes a continuous function. Make the decision "continuous" or "not continuous" in each of the cases a)-e) and point out any possible x values, which need special attention.

a) (2 points) $f(x) = \cos(\sin(5/x^2))$.

b) (2 points) $f(x) = x^2 \sin(5/x^2)$.

c) (2 points) $f(x) = \frac{\sin^2(x)}{x^2}$

d) (2 points) $f(x) = ||3x| - 4x|$

e) (2 points) $f(x) = 3x/|4x|$

Problem 5) Derivatives (10 points)

Do the required computations for the following functions. In each case, indicate which differentiation rule you use.

a) (2 points) For $f(x) = \sin(x)e^x$, write down $T(x) = x - f(x)/f'(x)$.

b) (2 points) For $f(x) = \log(\sin(x))$, write down $f''(x)$.

c) (2 points) For $f(x) = e^{(e^{\sin(x)})}$, write down $f'(x)$.

d) (2 points) For $f(x) = x^x$, write down $f'(x)$.

e) (2 points) Demonstrate how one can obtain the derivative of $\arcsin(x)$ with the help of the chain rule.

Problem 6) Limits (10 points)

Find the limits $\lim_{x \to 0} f(x)$ for the following functions. As always give details and indicate what method you are using.

a) (2 points) $f(x) = \frac{1-\exp(4x)}{1-\exp(3x)}$.

b) (2 points) $f(x) = \frac{1-x^2}{\cos(3x)}$.

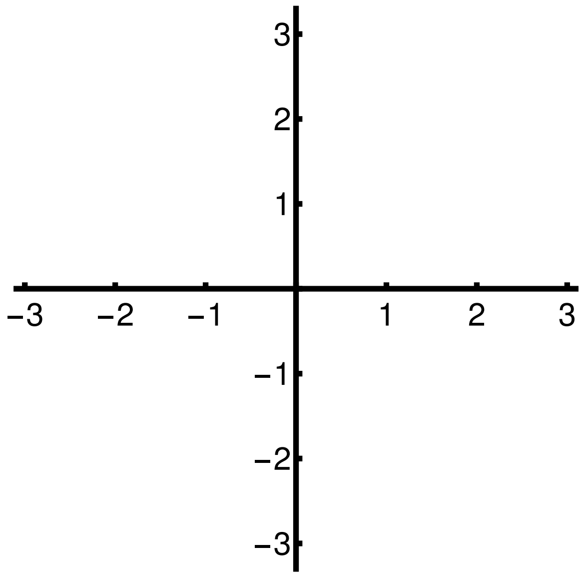
c) (2 points) $f(x) = \frac{3x}{\log(1+4x)}$.

d) (2 points) $f(x) = \frac{\log(x^3)}{\log(x^2)}$.

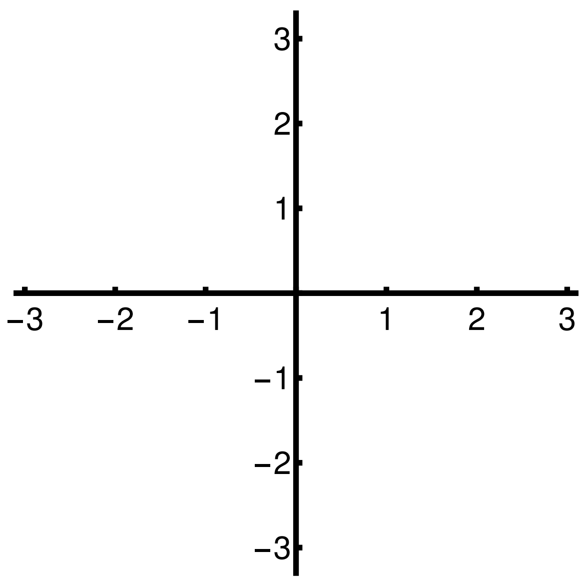
e) (2 points) $f(x) = x(\log(7x))^2$.

Problem 7) Functions (10 points)

a) (5 points) Draw the graph of the function $f(x) = 1 + \sin(3x)$ on $[\pi, \pi]$. Mark the roots, the maxima, minima and inflection points. On paper, please copy the template below exactly!



b) (5 points) Draw the graph of $f(x) = 1/\log|x|$ on $[-3, 3]$. Make sure to indicate what the values at $x = -1, x = 0, x = 1$ are, and draw any possible asymptotes.



Problem 8) Extrema (10 points)

a) (3 points) Find all the critical points of the function

$$f(x) = x^5 - 5x + 7 .$$

b) (3 points) Classify the critical points using the second derivative test.

c) (2 points) Find the global maximum and global minimum on $[-2, 2]$.

d) (1 point) Which mathematician proved that a global maximum and a global minimum exists for a continuous function on $[-2, 2]$?

e) (1 point) What is the name of that theorem on global maxima or minima?

Problem 9) Algebra (10 points, 2 points each)

Simplify the following expressions. For example, $3^x 3^y$ can be written fewer letters as 3^{x+y} or $\sin^2(x)/\sin(x)$ can be simplified as $\sin(x)$. As usual $\log(x) = \ln(x)$ is the **natural log**. Writing $\ln(x)$ instead of $\log(x)$ does not count as a simplification! In the second column, fill in the derivative $f'(x)$ of the expression $f(x)$

	Expression	Simplified expression	Derivative
a)	$\arccos(\cos(x^2))$		
b)	$3^{(4x/\log(3))}$		
c)	$\log(x^3)/(\cos^2(x)+\sin^2(x))$		
d)	$\cos(3x)\tan(3x)$		
e)	$\frac{(2x)^3}{(3x)^4}$		

3/5/2021: First hourly Practice A

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False ··· but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 6 AM on March 6th.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F The function $f(x) = \log(x)$ has a root at $x = 1$.

2) T F $f(x) = x^2 + \exp(-x^2)$ has a root on the real axes.

3) T F $\cos(\pi) = -1$.

4) T F $f(x) = \sin(|x|)$ is differentiable everywhere.

5) T F The chain rule assures that $\frac{d}{dx} \sin(x^5) = \cos(x) 5x^4$.

6) T F The function $f(x) = \exp(-\sin(x))$ is continuous everywhere.

7) T F $\sinh(x) = (e^x - e^{-x})/2$ is positive everywhere.

8) T F $\cot(x)$ is the inverse function of $\tan(x)$.

9) T F If $f(x)$ is differentiable at 0, then $f(x)^2$ is differentiable at 0.

10) T F The function $x^3/(1 + x^3)$ defines an indefinite form at $x = \infty$.

11) T F $\sin(3\pi/4) = -1$.

12) T F $f(x) = \tan(x)$ has a vertical asymptote at $x = 0$.

13) T F e^{x^2} takes the value π at some point.

14) T F If $f(x) = x^2$, then $Df(x) = f(x + 1) - f(x) = 2x + 1$.

15) T F The intermediate value theorem implies Rolles theorem.

16) T F The function $\sin(1/x)$ is continuous everywhere.

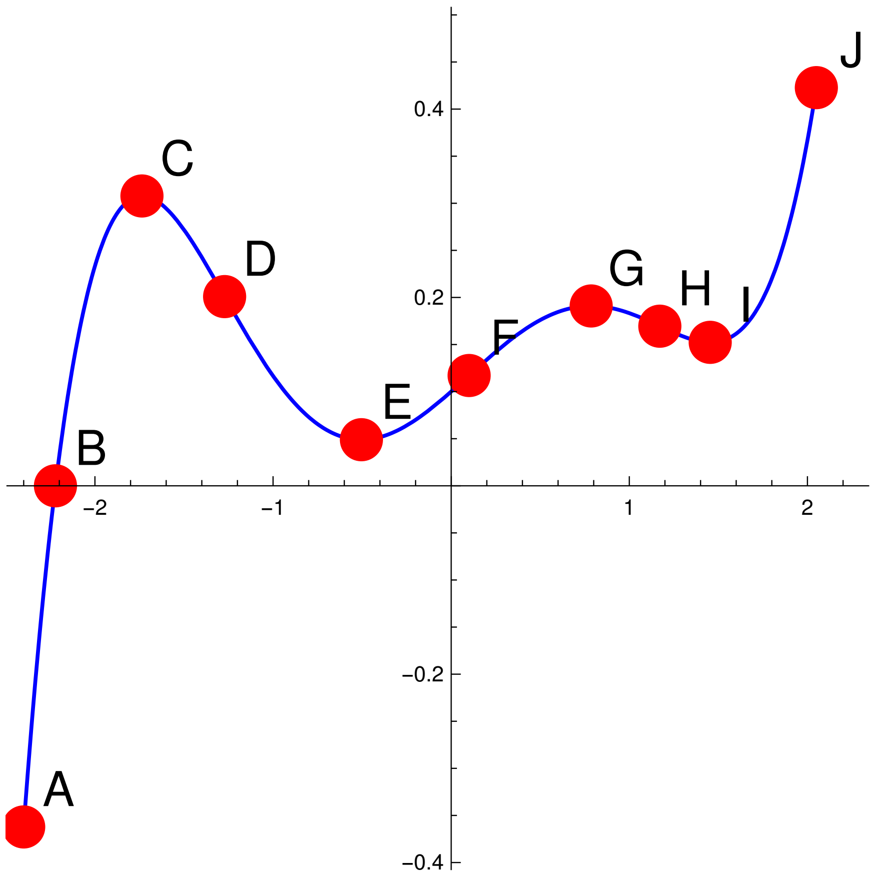
17) T F $\frac{d}{dx} \arctan(x) = 1/(1 + x^2)$.

18) T F $\frac{d}{dx} \log(3 + x) = 3/(3 + x)$.

19) T F A continuous function on $[0, 1)$ has at least one maximum.

20) T F The derivative of f/g is $(fg' - f'g)/g^2$.

Problem 2) Choice problem (10 points) No justifications are needed.



a) (2 points) List the points A-J which are roots of f.

b) (2 points) List the points A-J are inflection points.

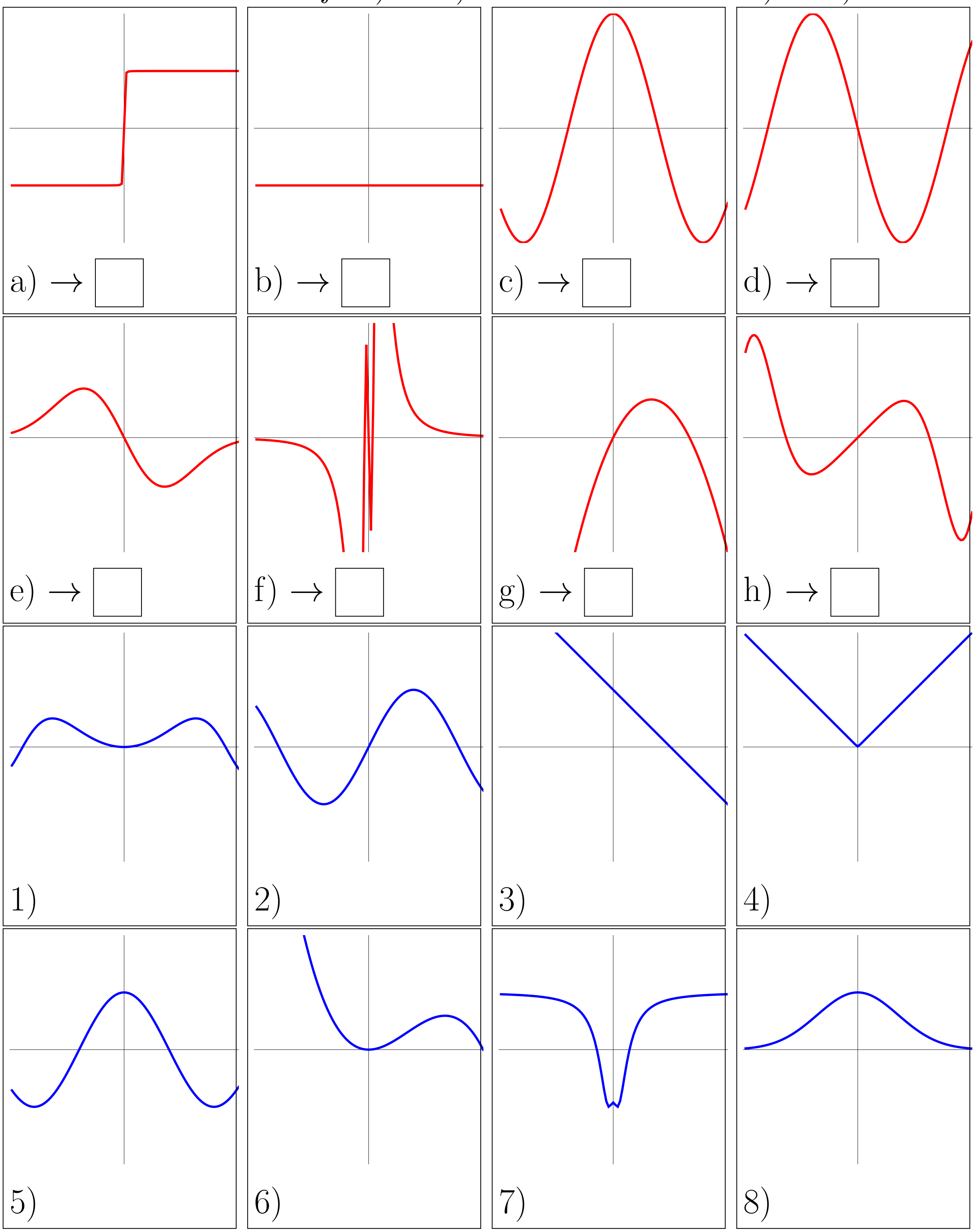
c) (2 points) List the points A-J that are local maxima.

d) (2 points) List the points A-J that are local minima.

e) (2 points) List the points A-J that are global maxima.

Problem 3) Matching problem (10 points) No justifications are needed.

Match the derivatives f' a) to h) with the functions 1) to 8).



Problem 4) Continuity (10 points)

Which of the following functions are continuous on $[-1, 1]$? As usual we extend the domain of definition to points, where a continuation is possible. In each case make the decision "continuous" or "not continuous" and point to the x value which needs special attention.

a) (2 points) $f(x) = \frac{x^6-1}{x^2-1}$

b) (2 points) $f(x) = \frac{\sin(\sin(x))}{\sin(\sin(\sin(x)))}$.

c) (2 points) $f(x) = \frac{\sin^2(x)}{2+\sin(x^2)}$

d) (2 points) $f(x) = \log|x|e^x$

e) (2 points) $f(x) = \frac{\sin(\tan(x))}{\sin(x)}$

Problem 5) Derivatives (10 points)

Find the derivatives of the following functions. In each case, indicate which differentiation rule you use.

a) (2 points) $f(x) = \frac{1}{1+e^x}$.

b) (2 points) $f(x) = \cos(x)\sin(x)$.

c) (2 points) $f(x) = \frac{1+x^3}{1+x^2}$.

d) (2 points) $f(x) = \arctan(\sin(x))$.

e) (2 points) $f(x) = \log(\log(5x))$.

Problem 6) Limits (10 points)

Find the limits $\lim_{x\to 0} f(x)$ for the following functions:

a) (2 points) $f(x) = \frac{\exp(7x)-1}{\exp(4x)-1}$.

b) (2 points) $f(x) = \frac{x-1}{x+1}$.

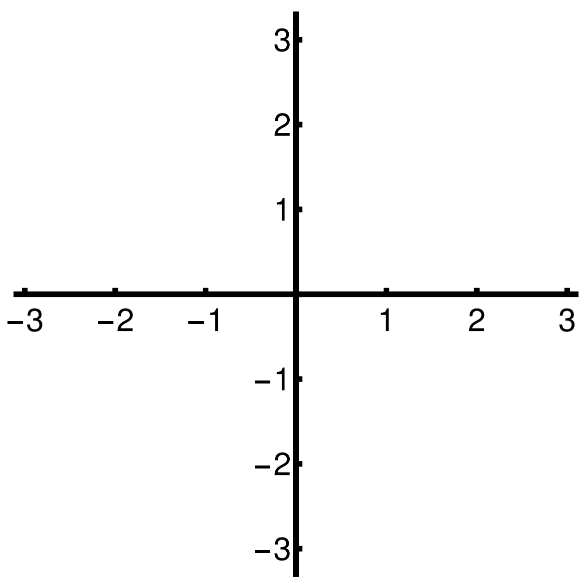
c) (2 points) $f(x) = \frac{\arctan(x)}{\sin(x)}$.

d) (2 points) $f(x) = \frac{\log(x^3)}{\log(x^2)}$.

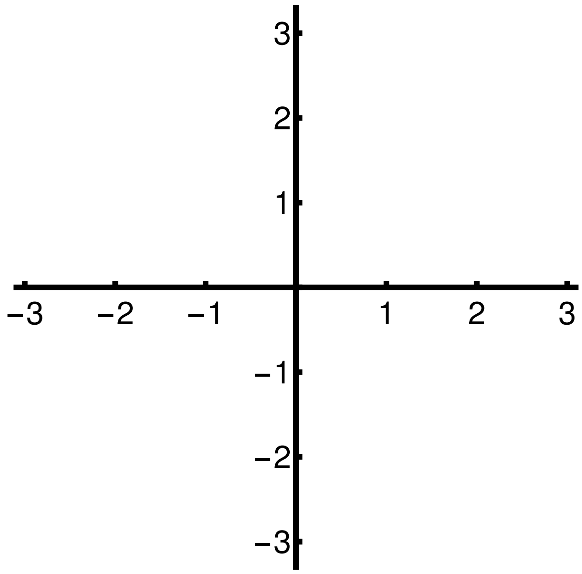
e) (2 points) $f(x) = \frac{\sin(3x)\sin(5x)}{\sin(7x)\sin(2x)}$.

Problem 7) Functions (10 points)

a) (5 points) Draw the graph of the natural log function $f(x) = \log|x| = \ln|x|$. Make sure you get the roots and asymptotes correct.



b) (5 points) Draw the graph of the arctan function $f(x) = \arctan(x)$. Make sure you get the roots and asymptotes correct.



Problem 8) Extrema (10 points)

a) (3 points) Find all the critical points of the function

$$f(x) = x^3 - 3x + 1 .$$

b) (3 points) Use the second derivative test to classify the critical points of f.

c) (2 points) On the interval $[-3, 3]$, where it the global maximum, and where is the global minimum?

d) (2 points) Which theorem assures that there is a global maximum and a global minimum on $[-3, 3]$?

Problem 9) Algebra rules (10 points, 2 points each)

a)	$(e^x)^y$	
b)	e^{x+y}	
c)	$\log(xy)$	
d)	$\frac{\tan(x)}{\sin(x)}$	
e)	$\frac{x^9}{x^3}$	

Choose from the following expressions.

- $1/\cos(x)$
- $\cos(x)/\sin^2(x)$.
- $e^x e^y$
- $\log(x+y)$
- $e^x + e^y$
- $e^{(xy)}$
- x^6
- $\log(x) - \log(y)$
- $e^{(x^y)}$
- $\log(x) + \log(y)$
- x^3
- $\cos(x)$

Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

3/5/2021: First hourly Practice B

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

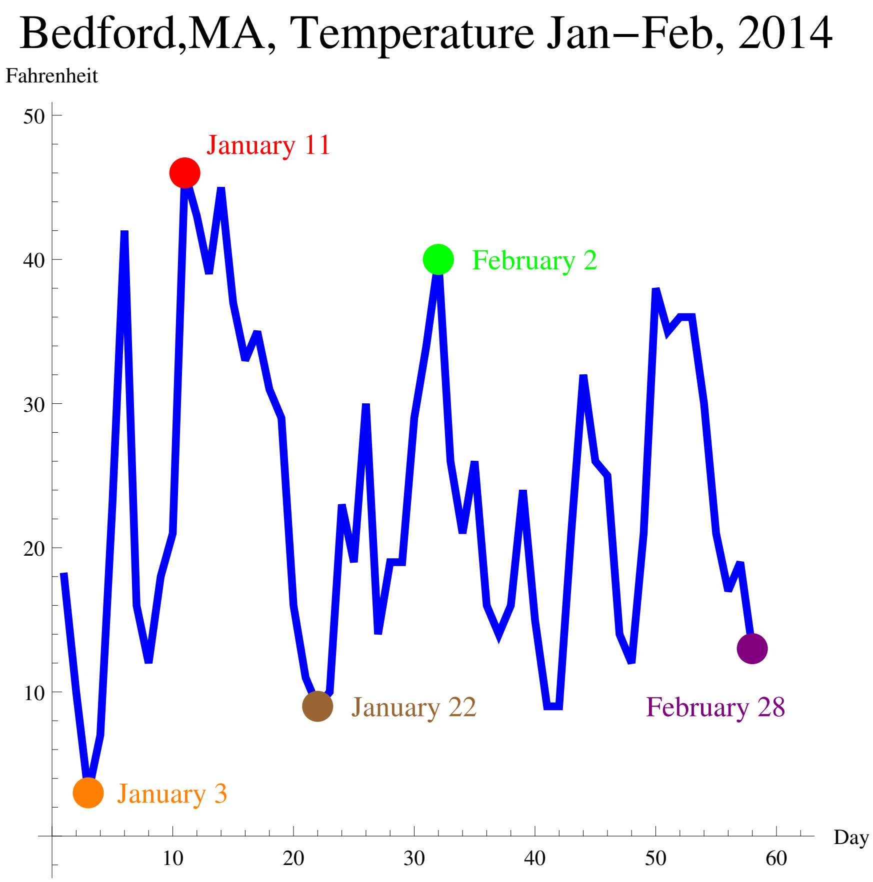
- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 6 AM on March 6th.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1)	T	F	The function $f(x) = \exp(-x^2) - 1$ has the root $x = 0$.
2)	T	F	If f is continuous function and odd then 0 is a root of f.
3)	T	F	$\log(\log(e)) = 0$, if log is the natural log.
4)	T	F	The chain rule assures that $\frac{d}{dx}\sin^2(x) = 2\cos(x)$.
5)	T	F	The function $f(x) = x^2/(1-x^2)$ is continuous everywhere on the real axes.
6)	T	F	The function $\arctan(x)$ is the inverse function of the function $\tan(x)$.
7)	T	F	The function $x^3/(1+x^3)$ defines an indefinite form at $x = 0$.
8)	T	F	$\cos(3\pi/2) = 0$.
9)	T	F	If a function f is continuous on $[-1,1]$ and $f(1) = 1, f(-1) = -1$, then there is $-1 < x < 1$, where $f(x) = 0$.
10)	T	F	The chain rule assures that $\frac{d}{dx}g(1/x) = -g'(1/x)/x^2$.
11)	T	F	We have $\lim_{x\to\infty}(2x+1)/(3x-1) = 2/3$.
12)	T	F	If 1 is a root of f, then $f'(x)$ changes sign at 1.
13)	T	F	If $f''(0) < 0$ and $f''(1) > 0$ then there is a point in $(0,1)$, where f has an inflection point.
14)	T	F	The intermediate value theorem assures that the equation $f(x) = x^2 - \cos(x) = 0$ has a root.
15)	T	F	The function $f(x) = x/\sin(x)$ is continuous everywhere if $f(0)$ is suitably defined.
16)	T	F	$f'(x) = 0$ and $f'''(0) < 0$ at $x = 0$ assures that f has a maximum at $x = 0$.
17)	T	F	If f is constant, then $f(x+h) - f(x)/h = 0$ for all $h > 0$.
18)	T	F	The quotient rule is $\frac{d}{dx}(f/g) = (f'(x)g'(x) - f(x)g(x))/(g'(x))^2$.
19)	T	F	$\sin(2\pi) + \tan(2\pi) = 0$.
20)	T	F	It is true that $e^{x\log(5)} = x^5$.

Problem 2) Matching problem (10 points) No justifications are needed.



A couple of years ago, the **polar vortex** ruled the weather in Boston. The above graph shows the temperatures of the first two months of 2014 measured at the **Hanscom field** in Bedford, MA. While temperatures are measured hourly, you can assume that temperature is a continuous function of time. Remember that "global maximum" includes being local too so that only one entry in each line of the table below needs to be checked.

a) (5 points) Check what applies, by checking one entry in each of the 5 dates.

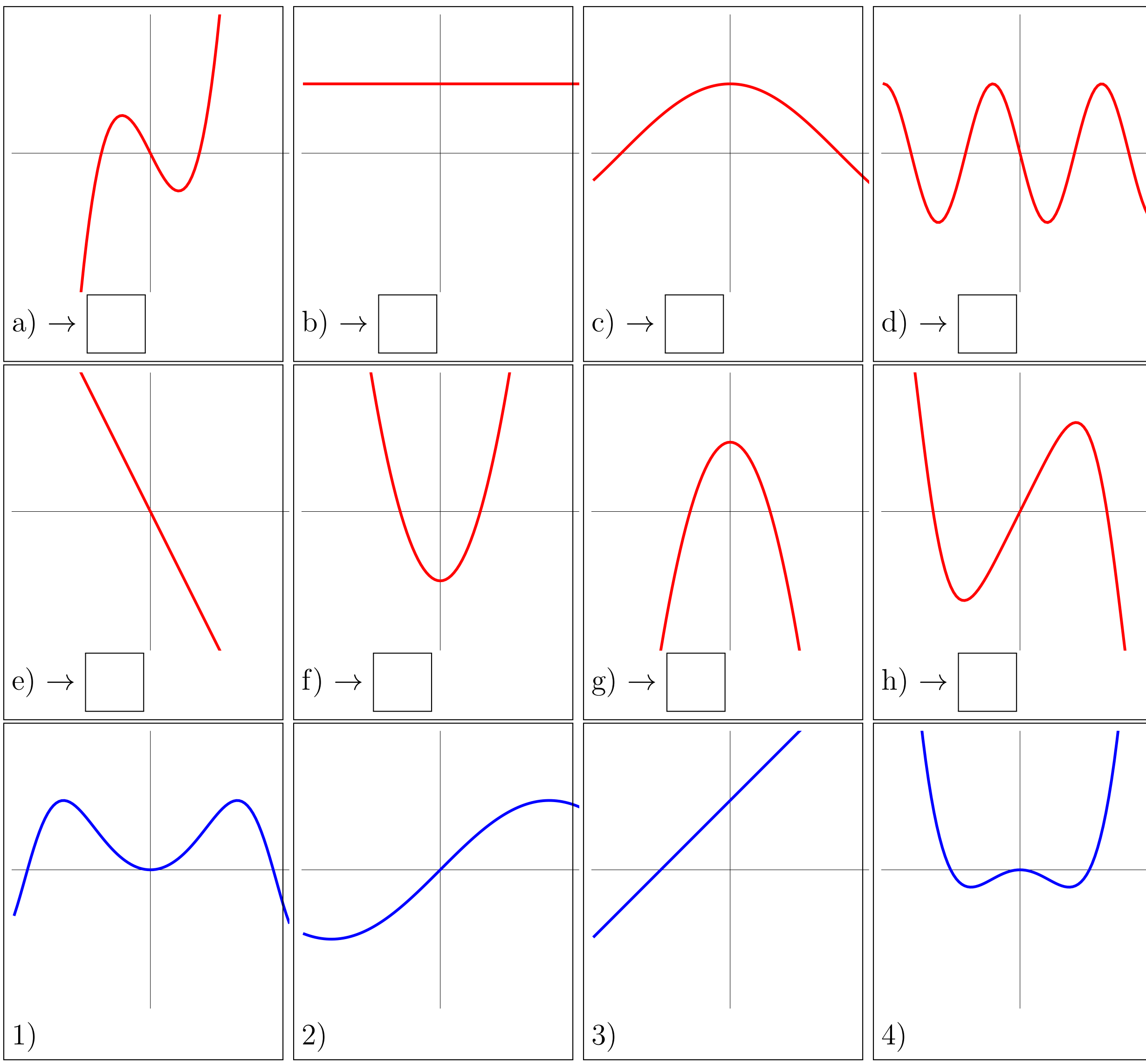
Date	local maximum	local minimum	global maximum	global minimum
January 3				
January 11				
January 22				
February 2				
February 28				

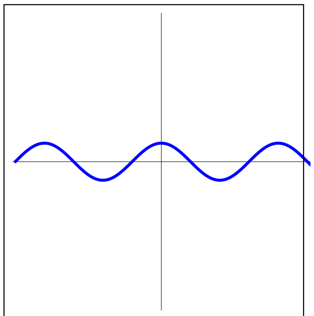
b) (2 points) Which theorem assures that on the closed interval $[0, 59]$ of 59 days, there is a global maximal temperature?

c) (3 points) Argue by citing a theorem why there is a time at which the temperature at Bedford was exactly 25 degree Fahrenheit.

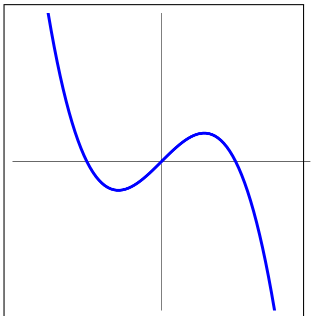
Problem 3) Matching problem (10 points) No justifications are needed.

In the first pictures, we see the first derivatives f'. Match them with the functions f in 1-8. Note that the functions above are the derivative and the functions below are the functions.

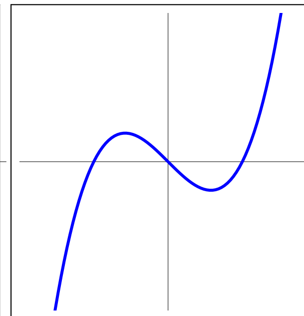




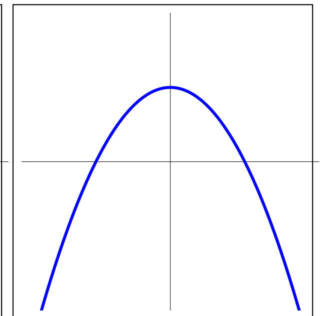
5)



6)



7)



8)

Problem 4) Continuity (10 points)

Each of the following functions has a point x_0, where the function is not defined. Find the limit $\lim_{x \to x_0} f(x)$ or state that the limit does not exist.

a) (2 points) $f(x) = \frac{1-2x^3}{1-x}$, at $x_0 = 1$.

b) (2 points) $f(x) = \sin(\sin(5x))/\sin(7x)$, at $x_0 = 0$.

c) (2 points) $f(x) = \frac{\exp(-3x)-1}{\exp(2x)-1}$, at $x_0 = 0$.

d) (2 points) $f(x) = \frac{2x}{\log(x)}$, at $x_0 = 0$.

e) (2 points) $f(x) = \frac{(x-1)^{10}}{(x+1)^{10}}$, at $x_0 = -1$.

Problem 5) Derivatives (10 points)

Find the derivatives of the following functions. If you use a differentiation rule, note which one you use.

a) (2 points) $f(x) = \sqrt{\log(x+1)}$.

b) (3 points) $f(x) = 7\sin(x^3) + \frac{\log(5x)}{x}$.

c) (3 points) $f(x) = \log(\sqrt{x}) + \arctan(x^3)$.

d) (2 points) $f(x) = e^{5\sqrt{x}} + \tan(x)$.

Problem 6) Limits (10 points)

Find the limits $\lim_{x \to 0} f(x)$ for the following functions:

a) (2 points) $f(x) = \frac{\exp(3x)-\exp(-3x)}{\exp(5x)-\exp(-5x)}$.

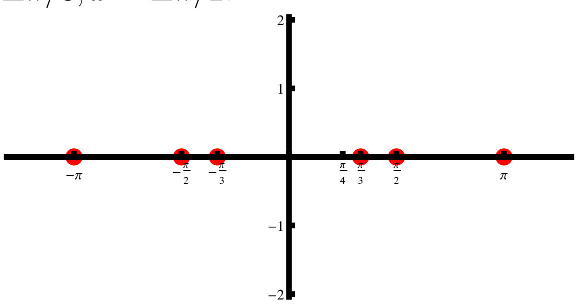
b) (3 points) $f(x) = \frac{\cos(3x)-1}{\sin^2(x)}$.

c) (3 points) $f(x) = [\arctan(x) - \arctan(0)]/x$.

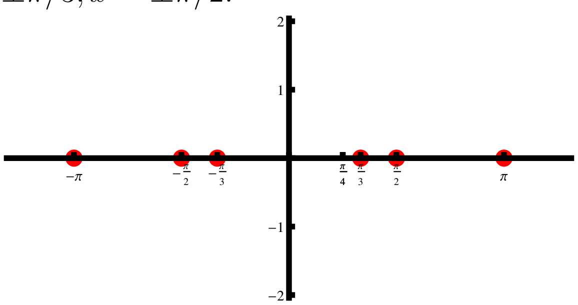
d) (2 points) $f(x) = \frac{\log(7x)}{\log(11x)}$.

Problem 7) Trig functions (10 points)

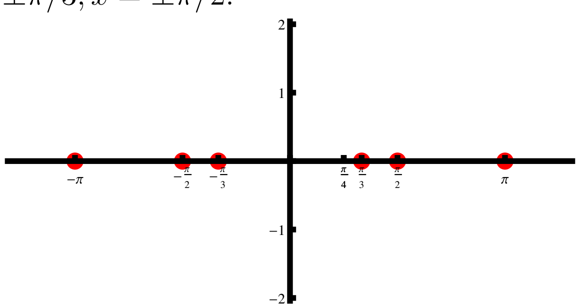
a) Draw the sin function and mark the values of $\sin(x)$ at $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$.



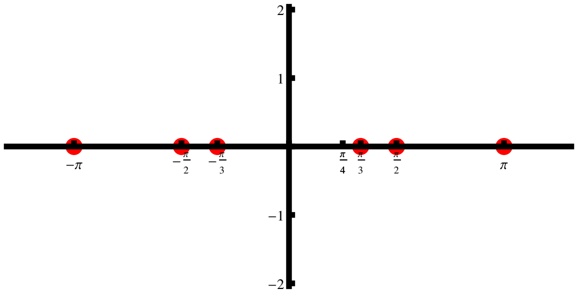
b) Draw the cos function and mark the values of $\cos(x)$ at $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$.



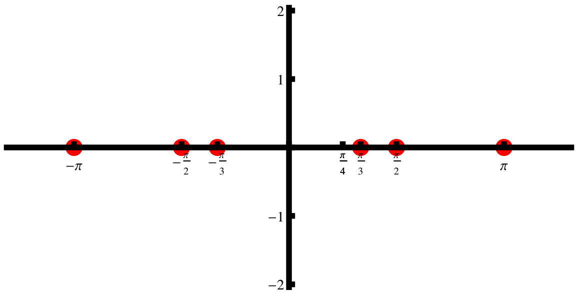
c) Draw the tan function and mark the values of $\tan(x)$ at $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$.



d) Draw the cot function and mark the values of $\cot(x)$ at $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$.



e) Draw the sinc function $f(x) = \sin(x)/x$ and mark the points $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2,$.



Problem 8) Extrema (10 points)

You got a batch of strong **Neodym magnets**. They are ring shaped. Assume the inner radius is x, the outer radius y is 1 and the height is $h = x$, we want to maximize the surface area $A = 2\pi(y-x)h + 2\pi(y^2 - x^2)$. This amount of maximizing

$$f(x) = 2\pi(1-x)x + 2\pi(1-x^2)$$

a) (2 points) Using that $f(x)$ is a surface area, on what interval $[a, b]$ needs f to be considered?

b) (3 points) Find the local maxima of f inside the interval.

c) (3 points) Use the second derivative test to verify it is a maximum.

d) (2 points) Find the global maximum on the interval.

Problem 9) Trig and Exponential functions (10 points)

Simplify the following terms. log denotes the natural log and $\log_{10}$ the log to the base 10. Each result in a)-c) is an integer or a fraction

a) (2 points) $\exp(\log(2)) + e^{3\log(2)}$

b) (2 points) $\log(1/e) + \exp(\log(2)) + \log(\exp(3))$.

c) (2 points) $\log_{10}(1/100) + \log_{10}(10000)$

d) (4 points) Produce the formula for $\arccos'(x)$ by taking the derivative of the identity

$$\cos(\arccos(x)) = x \ .$$

Your answer should be simplified as we did when deriving the derivatives of arcsin, arctan in class or when you derived the derivative of arccot and arcsinh, arccosh in the homework.

3/5/2021: First hourly Practice C

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 6 AM on March 6th.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F If f is concave up on $[0,1]$ and concave down on $[1,2]$ then 1 is an inflection points.

2) T F The function $f(x) = \exp(x)$ has the root $x = 1$.

3) T F $\log(\exp(1)) = 1$, if log is the natural log and $\exp(x) = e^x$ is the exponential function.

4) T F The chain rule assures that $d/dx f(f(x)) = f'(f(x))f'(x)$.

5) T F The function $x^2/(1+x^2)$ is continuous everywhere on the real axes.

6) T F The function $\cot(x)$ is the inverse of the function $\tan(x)$.

7) T F The function $\tan(x)/\log|x|$ defines an indefinite form at 0.

8) T F $\cos(\pi/2) = 1/2$.

9) T F If a function f is differentiable on $[-1,1]$, then there is a point x in that interval where $f'(x) = 0$.

10) T F The chain rule assures that $d/dx(g(x^2)) = 2xg'(x^2)$.

11) T F We have $\lim_{x\to\infty}((x^2+1)/x^2) = 1$

12) T F An inflection point is a point, where the function $f'(x)$ changes sign.

13) T F If $f''(-2) > 0$ then f is concave up at $x = -2$.

14) T F The intermediate value theorem assures that the continuous function $x + \sin(x) = 0$ has a root.

15) T F We can find a value b and define $f(0) = b$ such that the function $f(x) = (x^{28}-1)/(x^2-1)$ is continuous everywhere.

16) T F If the third derivative $f'''(x)$ is negative and $f''(x) = 0$ then f has a local maximum at x.

17) T F If $f(x) = x^2$ then $Df(x) = f(x+1) - f(x)$ has a graph which is a line.

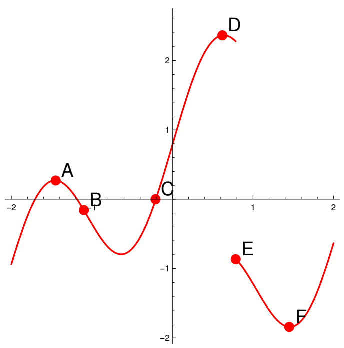
18) T F The quotient rule is $d/dx(f/g) = f'(x)/g'(x)$.

19) T F With $Df(x) = f(x+1) - f(x)$, we have $D(1+a)^x = a(1+a)^x$.

20) T F It is true that $\log(5)e^x = e^{x\log(5)}$ if $\log(x)$ is the natural log.

Problem 2) Matching problem (10 points) No justifications are needed.

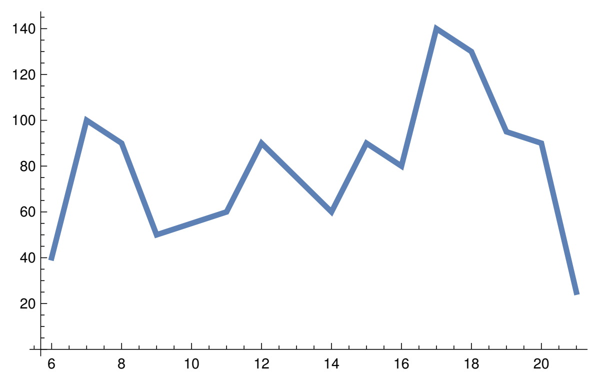
a) (6 points) You see the graph of a function $f(x)$ defined on $[-2, 3]$. Various points $(x, f(x))$ are marked. Match them:



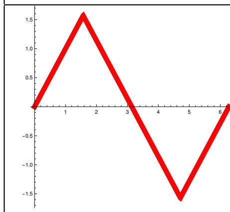
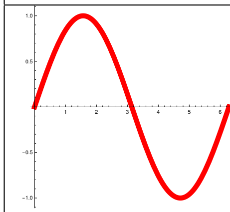
Point x is	Fill in A-F
Local maximum	
Root	
Inflection point	
Discontinuity	
Global maximum	
Local minimum	

b) (2 points) the **Harvard recreation** publishes regularly a graph of a function $f(x)$ which shows the number of people at the **Mac** gym as a function of time. At 5 o'clock, there are in average 120 visitors, at 9 in the morning, there are 60 people working out. By the intermediate value theorem, there must be a moment at which exactly $\pi^4 = 97.5...$ visitors are present. This is obviously nonsense. Where is the flaw?

Reason	Check one
No differentiability	
Statistical glitch	
No Continuity	
Inaccurate data	

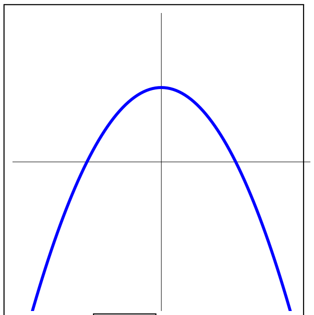


c) (2 points) In front of the **"Class of 1959 Chapel"** at the Harvard business school is an amazing clock: a marble tower contains a steel pole and a large bronze ball which moves up and down the pole indicating the time of the day. As the ball moves up and down the pole, lines with equal distance on the tower indicate the time. At noon, the sphere is at the highest point. At midnight it is at the bottom. It moves the same distance in each hour. If we plot the hight of the sphere as a function of time, which graph do we see?

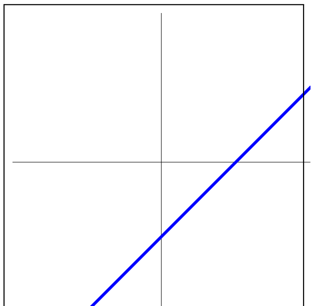
The height function	Check which applies
	
	

Problem 3) Matching problem (10 points) No justifications are needed.

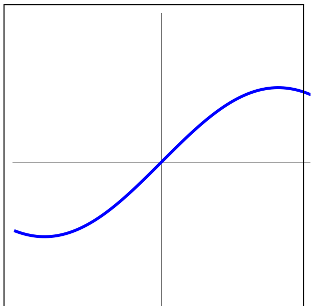
Match the graph of the functions f in $a) - h)$ with the derivatives f' in 1)-8).



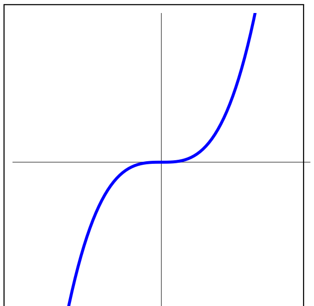
a) $\rightarrow$ ☐



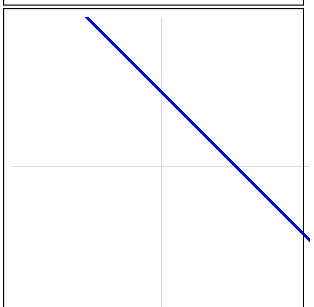
b) $\rightarrow$ ☐



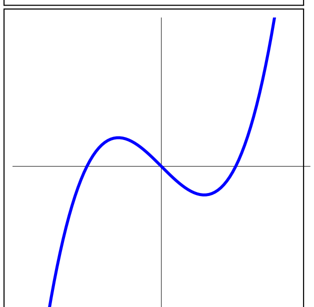
c) $\rightarrow$ ☐



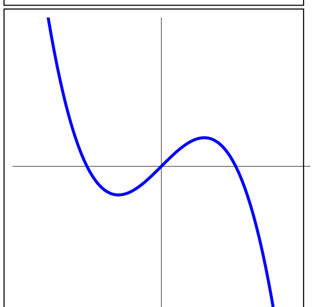
d) $\rightarrow$ ☐



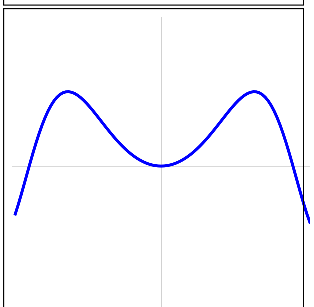
e) $\rightarrow$ ☐



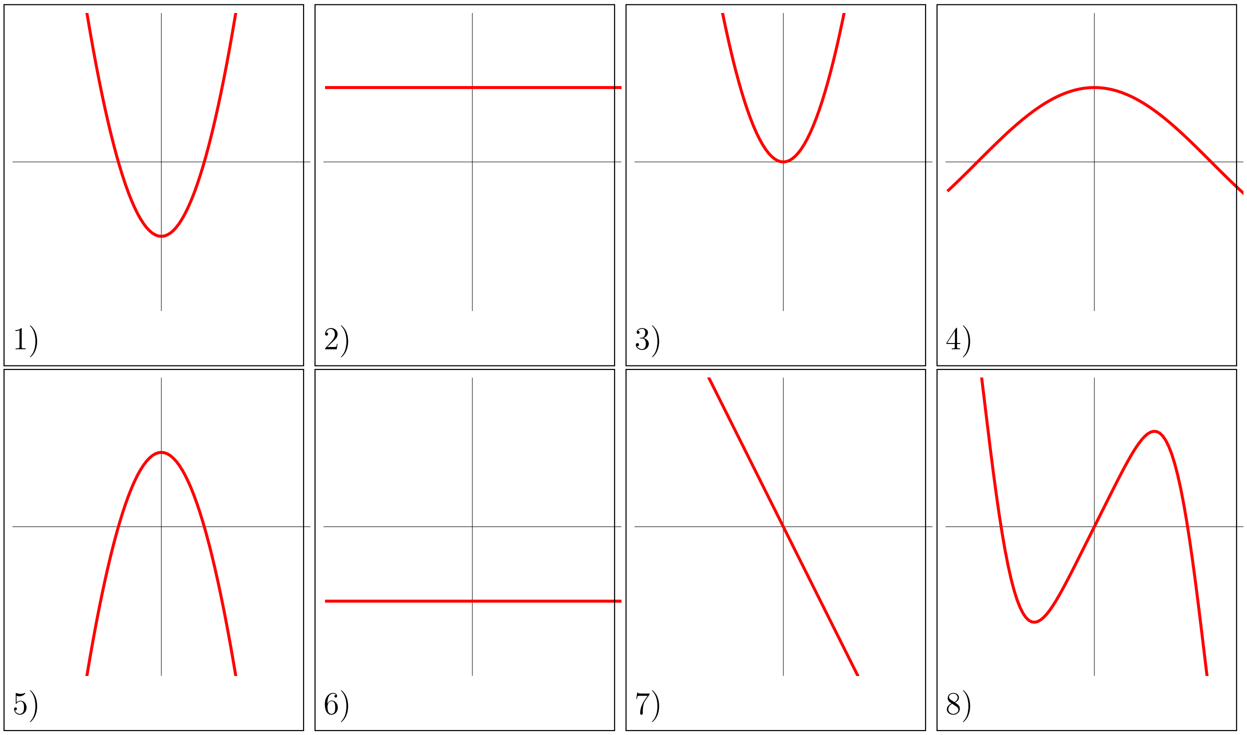
f) $\rightarrow$ ☐



g) $\rightarrow$ ☐



h) $\rightarrow$ ☐



Problem 4) Continuity (10 points)

Each of the following functions has a point x_0, where the function is not defined. Find the limit $\lim_{x \to x_0} f(x)$ or state that the limit does not exist.

a) (2 points) $f(x) = \frac{x^3-8}{x-2}$, at $x_0 = 2$

b) (2 points) $f(x) = \sin(\sin(\frac{1}{x})) - \tan(x)$, at $x_0 = 0$

c) (2 points) $f(x) = \frac{\cos(x)-1}{x^2}$, at $x_0 = 0$

d) (2 points) $f(x) = \frac{\exp(x)-1}{\exp(5x)-1}$, at $x_0 = 0$

e) (2 points) $f(x) = \frac{x-1}{x}$, at $x_0 = 0$

Problem 5) Derivatives (10 points)

Find the derivatives of the following functions. If you use a differentiation rule, note which one you use.

a) (2 points) $f(x) = \sin(7x) + (1 + x^2)$.

b) (2 points) $f(x) = \frac{\sin(7x)}{(1+x^2)}$.

c) (2 points) $f(x) = \sin(7 + x^2)$.

d) (2 points) $f(x) = \sin(7x)(1 + x^2)$.

e) (2 points) $f(x) = \sin(7x)^{(1+x^2)}$

Problem 6) Limits (10 points)

Find the limits $\lim_{x \to 0} f(x)$ for the following functions f at $x = 0$:

a) (2 points) $f(x) = \frac{(1-\exp(11x))}{(1-\exp(3x))}$

b) (2 points) $f(x) = \frac{\sin(\sin(5x))}{\sin(7x)}$

c) (2 points) $f(x) = \frac{\log(x)}{\log(5x)}$

d) (2 points) $f(x) = \frac{x^2 \cos(x)}{\sin^2(x)}$

e) (2 points) $f(x) = \frac{(1+1/x^2)}{(1-1/x^2)}$

Problem 7) Trig functions (10 points)

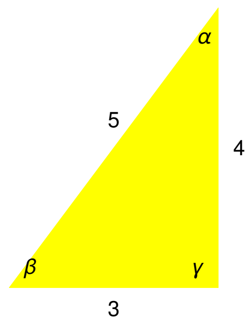
A triangle with side lengths $3, 4, 5$ has a right angle. Let $\alpha < \beta < \gamma$ denote the angles ordered by size.

a) (4 points) What are the numerical values of $\cos(\alpha), \cos(\beta), \cos(\gamma), \sin(\gamma)$?

b) (2 points) Find the numerical value of $\tan(\alpha)$ and $\cot(\alpha)$.

The next problem is independent of the previous two.

c) (4 points) Find the derivative of the inverse function of $\arcsin(x)$ by starting with the identity $x = \sin(\arcsin(x))$. Your derivation of $\arcsin'(x)$ should convince somebody who does not know the identity already.



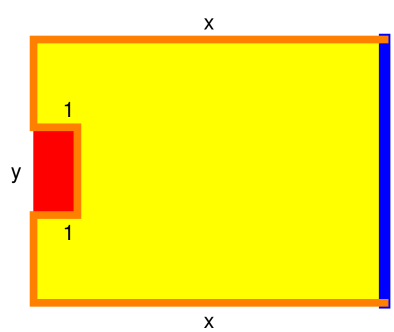
Problem 8) Extrema (10 points)

A tennis field of width x and length y contains a fenced referee area of length 2 and width 1 within the field and an already built wall. The circumference a fence satisfies $2x + y + 2 = 10$, (an expression which still can be simplified). We want to maximize the area $xy - 2$.

a) (2 points) On which interval $[a, b]$ does the variable x make sense? Find a function $f(x)$ which needs to be maximized.

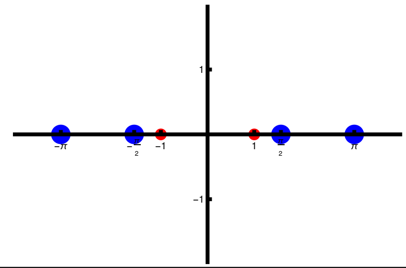
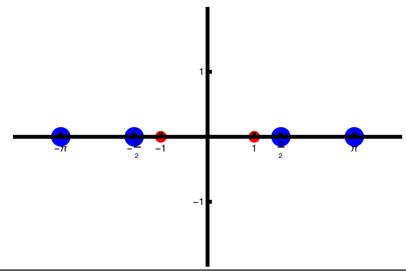
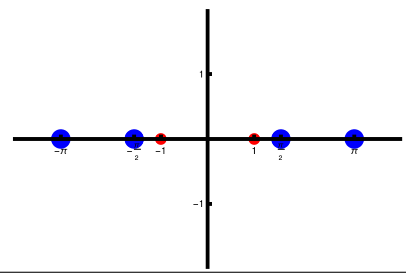
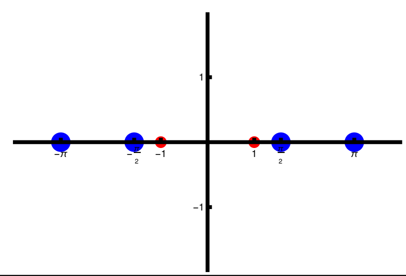
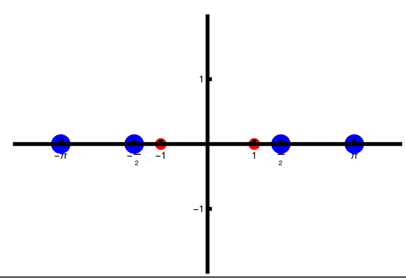
b) (6 points) Find the local maximum of x and check it with the second derivative test.

c) (2 points) What is the global maximum of f on $[a, b]$?



Problem 9) Trig functions (10 points)

a) In the following five problems, find the numerical value and then draw the graph of the function.

Problem	Answer	Graph
A) (2 points) What is $\sin(\pi/3)$? Plot $\sin(x)$.		
B) (2 points) What is $\cos(5\pi/2)$? Plot $\cos(x)$.		
C) (2 points) Find $\arctan(1)$ Plot $\arctan(x)$.		
D) (2 points) What is $\log(1)$ Plot $\log\|x\|$.		
E) (2 points) What is $\arcsin(\sqrt{3}/2)$. Plot $\arcsin(x)$		

b) Simplify the following terms. log denotes the natural log and $\log_{10}$ denotes the log to the base 10. All answers are integers.

A) (2 points) $\exp(\log(2))$

B) (2 points) $e^{\log(2)3}$

C) (2 points) $\log(\log(e))$

D) (2 points) $\exp(\log(2) + \log(3))$

E) (2 points) $\log_{10}(10000)$

Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

3/5/2021: First hourly Practice D

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False ··· but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 6 AM on March 6th.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points) No justifications are needed.

1) T F The function $\cot(x)$ is the inverse of the function $\tan(x)$.

2) T F We have $\cos(x)/\sin(x) = \cot(x)$

3) T F $\sin(3\pi/2) = -1$.

4) T F The function $f(x) = \sin(x)/x$ has a limit at $x = 0$.

5) T F For the function $f(x) = \sin(\sin(\exp(x)))$ the limit $\lim_{h\to 0}[f(x+h)-f(x)]/h$ exists.

6) T F If a differentiable function $f(x)$ satisfies $f'(3) = 3$ and is f' is odd then it has a critical point.

7) T F The l'Hospital rule assures that the derivative satisfies $(f/g)' = f'/g'$.

8) T F The intermediate value theorem assures that a continuous function has a derivative.

9) T F After healing, the function $f(x) = (x+1)/(x^2-1)$ is continuous everywhere.

10) T F If f is concave up on $[1, 2]$ and concave down on $[2, 3]$ then 2 is an inflection point.

11) T F There is a function f which has the property that its second derivative f'' is equal to its negative f.

12) T F The function $f(x) = [x]^4 = x(x-h)(x-2h)(x-3h)$ has the property that $Df(x) = 4[x]^3 = 4x(x-h)(x-2h)$, where $Df(x) = [f(x+h) - f(x)]/h$.

13) T F The quotient rule is $d/dx(f/g) = (f'g - fg')/g^2$ and holds whenever $g(x) \neq 0$.

14) T F The chain rule assures that $d/dx f(g(x)) = f'(g(x)) + f(g'(x))$.

15) T F If f and g are differentiable, then $(3f + g)' = 3f' + g'$.

16) T F The function $(e^x - 1)/(\cos(x) - 1)$ defines an indefinite form at 0.

17) T F One can rotate a four legged table on an arbitrary surface such that all four legs are on the ground.

18) T F The fundamental theorem of calculus relates integration S with differentiation D. The result is $DSf(x) = f(x), SDf(x) = f(x) - f(0)$.

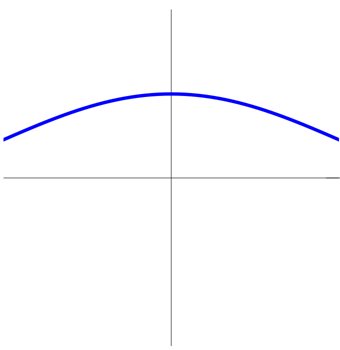
19) T F The product rule implies $d/dx(f(x)g(x)h(x)) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$.

20) T F Euler and Gauss are the founders of infinitesimal calculus.

Problem 2) Matching problem (10 points) No justifications are needed.

Match the following functions with their graphs.

Function	Fill in 1-8
$x^2 - x$	
$\exp(-x)$	
$\sin(3x)$	
$\log(\|x\|)$	
$\tan(x)$	
$1/(2 + \cos(x))$	
$x - \cos(6x)$	
$\sin(3x)/x$	



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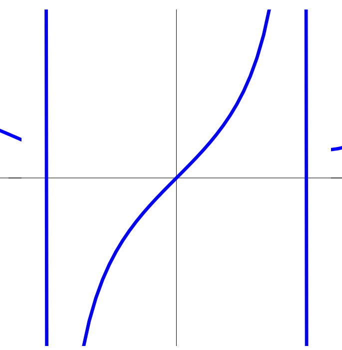

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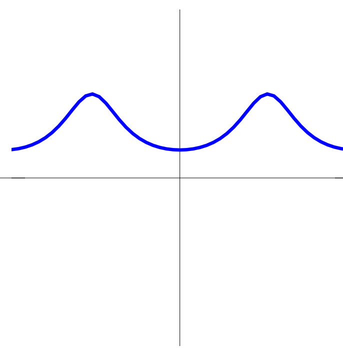

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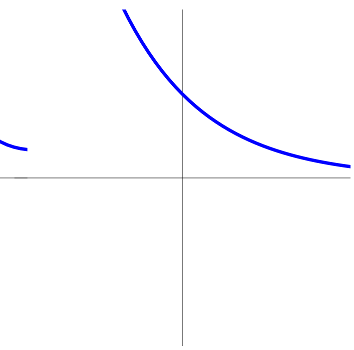

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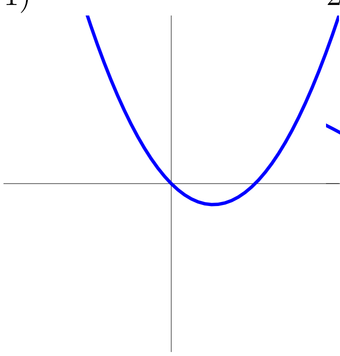

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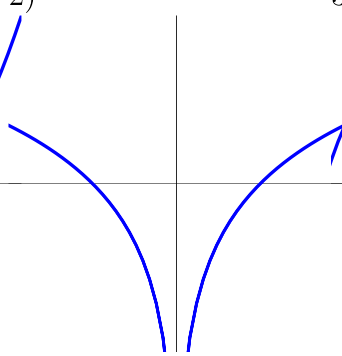

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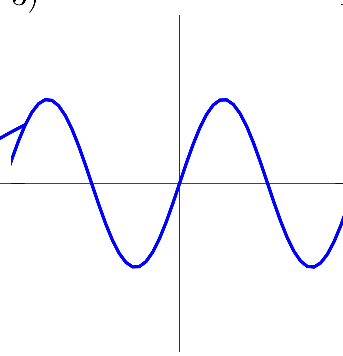

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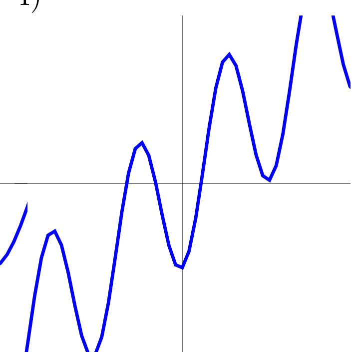
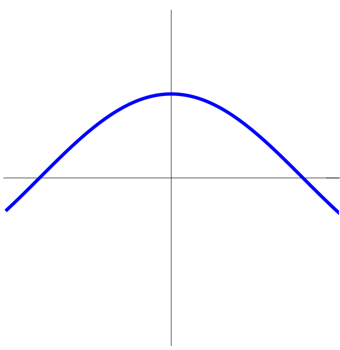


8)

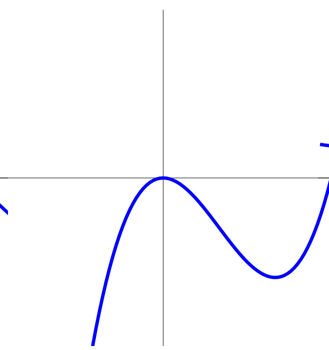
Problem 3) Matching problem (10 points) No justifications are needed.

Match the following functions with their derivatives.

Function	Fill in the numbers 1-8
graph a)	
graph b)	
graph c)	
graph d)	
graph e)	
graph f)	
graph g)	
graph h)	



a)



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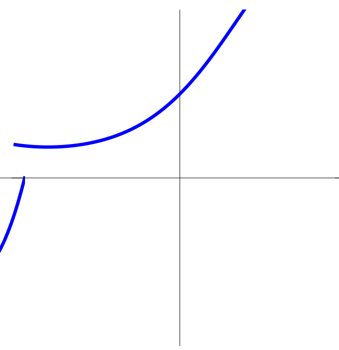

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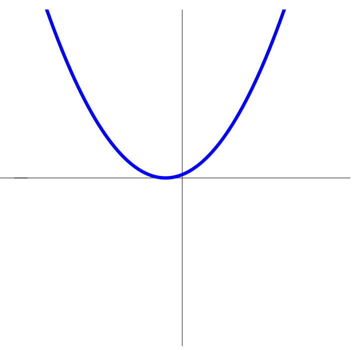

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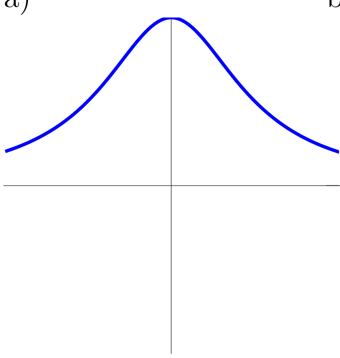

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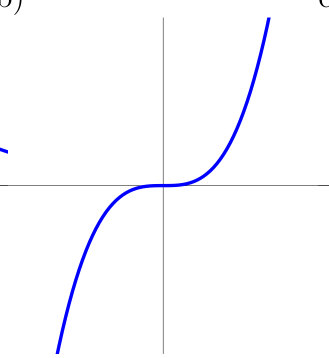

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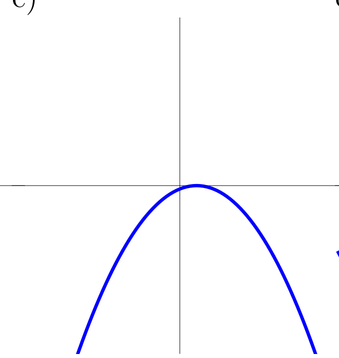

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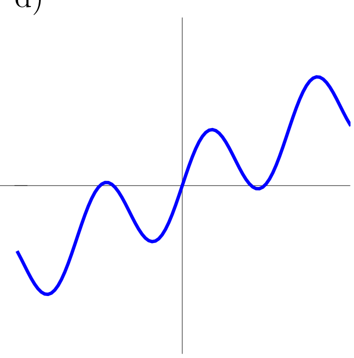

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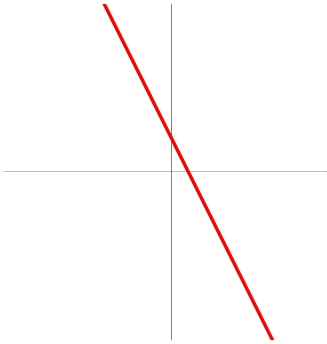
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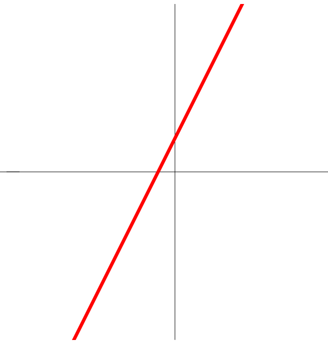
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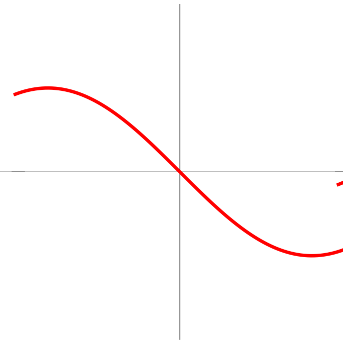
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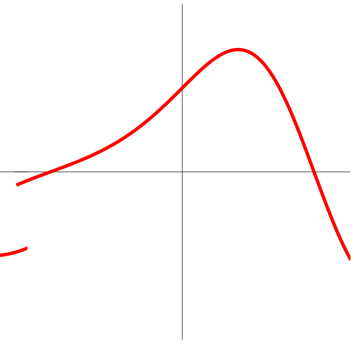
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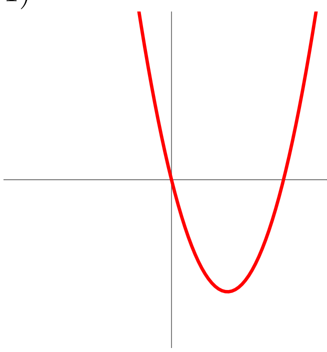
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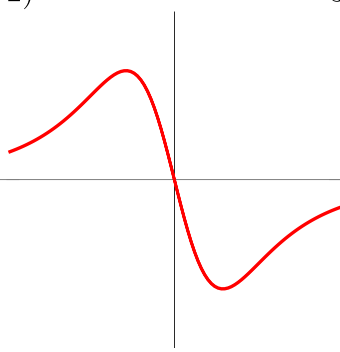
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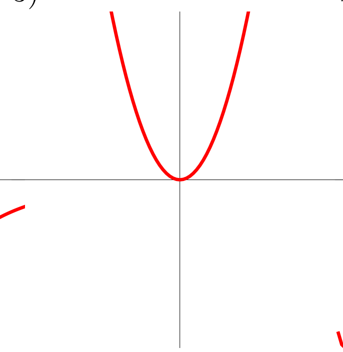
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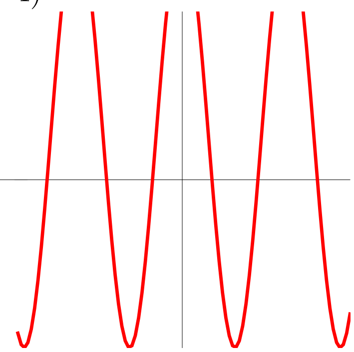

8)

Problem 4) Functions (10 points) No justifications are needed

Match the following functions with simplified versions. In each of the rows, exactly one of the choices A-C is true.

Function	Choice A	Choice B	Choice C	Enter A-C
$\frac{x^4-1}{x-1}$	$1+x+x^2+x^3$	$1+x+x^2$	$1+x+x^2+x^3+x^4$	
2^x	$e^{2\log(x)}$	$e^{x\log(2)}$	$2^{e\log(x)}$	
$\sin(2x)$	$2\sin(x)\cos(x)$	$\cos^2(x)-\sin^2(x)$	$2\sin(x)$	
$(1/x)+(1/(2x))$	$1/(x+2x)$	$3/(2x)$	$1/(x+2x)$	
e^{x+2}	$e^x e^2$	$2e^x$	$(e^x)^2$	
$\log(4x)$	$4\log(x)$	$\log(4)\log(x)$	$\log(x)+\log(4)$	
$\sqrt{x}^3$	$x^{3/2}$	$x^{2/3}$	$3\sqrt{x}$	

Problem 5) Roots (10 points)

Find the roots of the following functions

a) (2 points) $7\sin(3\pi x)$

b) (2 points) $x^5 - x$.

c) (2 points) $\log|ex|$.

d) (2 points) $e^{5x} - 1$

e) (2 points) $8x/(x^2+4) - x$.

Problem 6) Derivatives (10 points)

Find the derivatives of the following functions:

a) (2 points) $f(x) = \cos(3x)/\cos(10x)$

b) (2 points) $f(x) = \sin^2(x)\log(1+x^2)$

c) (2 points) $f(x) = 5x^4 - 1/(x^2+1)$

d) (2 points) $f(x) = \tan(x) + 2^x$

e) (2 points) $f(x) = \arccos(x)$

Problem 7) Limits (10 points)

Find the limits $\lim_{x \to 0} f(x)$ of the following functions:

a) (2 points) $f(x) = (x^6 - 3x^2 + 2x)/(1 + x^2 - \cos(x))$.

b) (2 points) $f(x) = (\cos(3x) - 1)/(\cos(7x) - 1)$.

c) (2 points) $f(x) = \tan^3(x)/x^3$.

d) (2 points) $f(x) = \sin(x)\log(x^6)$

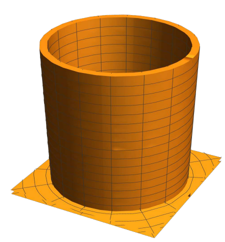
e) (2 points) $f(x) = 4x(1 - x)/(\cos(x) - 1)$.

Problem 8) Extrema (10 points)

a) (5 points) Find all local extrema of the function $f(x) = 30x^2 - 5x^3 - 15x^4 + 3x^5$ on the real line.

Problem 9) Extrema (10 points)

(10 points) A cup of height h and radius r has the volume $V = \pi r^2 h$. Its surface area is $\pi r^2 + \pi r h$. Among all cups with volume $V = \pi$ find the one which has minimal surface area. Find the global minimum.



3/5/2021: First hourly Practice E

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 6 AM on March 6th.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

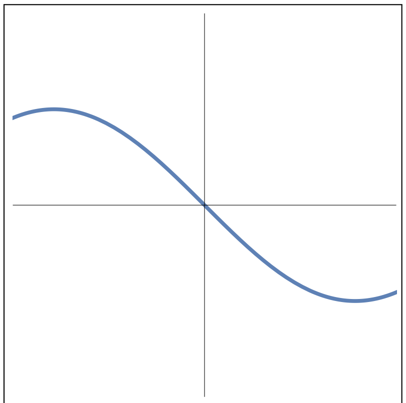
1)	T	F	The function arcsin(x) is defined as $1/\sin(x)$.
2)	T	F	The function $f(x) = \sin(1/x^2)$ can be defined at 0 so that it becomes a continuous everywhere on the real line.
3)	T	F	The function $x/\sin(x)$ can be defined at $x = 0$ so that it becomes a continuous function on the real line.
4)	T	F	The function $f(x) = \sin^2(x)/x^2$ has the limit 1 at $x = 0$.
5)	T	F	The function $f(x) = 1/\log\|x\|$ has the limit 1 at $x = 0$.
6)	T	F	The function $f(x) = (1+h)^{x/h}$ has the property that $Df(x) = [f(x+h) - f(x)]/h = f(x)$.
7)	T	F	$\cos(3\pi/2) = 1$.
8)	T	F	If a function f is continuous on the interval $[3, 10]$, then it has a global maximum on this interval.
9)	T	F	The reciprocal rule assures that $d/dx(1/g(x)) = 1/g(x)^2$.
10)	T	F	If $f(0) = g(0) = f'(0) = g'(0) = 0$ and $g''(0) = f''(0) = 1$, then $\lim_{x\to 0}(f(x)/g(x)) = 1$
11)	T	F	An inflection point is a point where the function $f''(x)$ changes sign.
12)	T	F	If $f''(x) > 0$ then f is concave up at x.
13)	T	F	The chain rule assures that $d/dx f(g(x)) = f'(x)g'(x)$.
14)	T	F	The function $f(x) = 1/x + \log(x)$ is continuous on the interval $[1, 2]$.
15)	T	F	The function $(e^x - 1)/\cos(x)$ defines an indefinite form at ∞.
16)	T	F	The graph of the function $f(x) = x/(1+x^2)$ has slope 1 at 0.
17)	T	F	There is a differentiable function for which $f'(0) = 0$ but for which 0 is not a local extremum.
18)	T	F	The second derivative test assures that $x = p$ is a local minimum if $f'(p) = 0$ and $f''(p) < 0$.
19)	T	F	The identity $(x^7 - 1)/(x - 1) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ holds for all $x \neq 1$.
20)	T	F	The slope of the tangent at a point $(x, f(x))$ of the graph of a differentiable function f is equal to $1/f'(x)$.

Problem 2) Matching problem (10 points) No justifications are needed.

Match the functions with the graphs. Naturally, only 10 of the 12 graphs will appear.

Function	Enter 1-12
$\cot(x)$	
$\cos(2x)$	
$2x$	
$\tan(x)$	
$\log(1/\|x\|)$	

Function	Enter 1-12
x^2	
$\exp(x)$	
$-\sin(x)$	
x^3	
$\mathrm{sinc}(x)$	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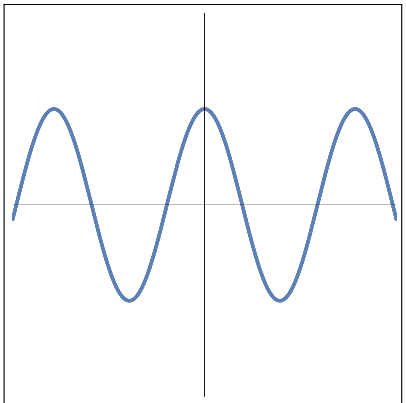
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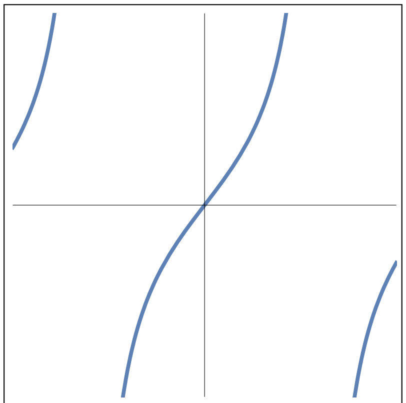
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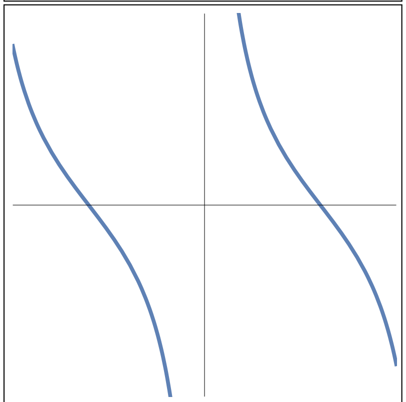
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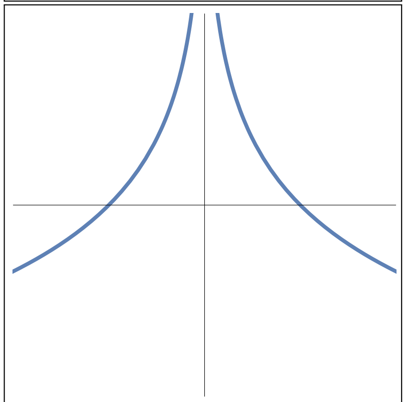
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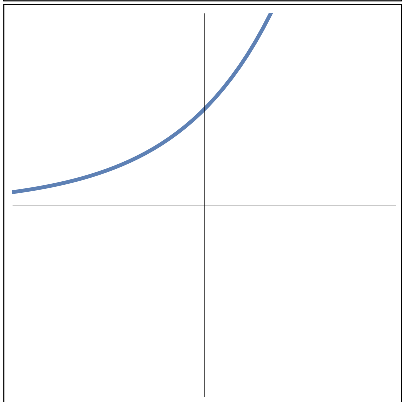
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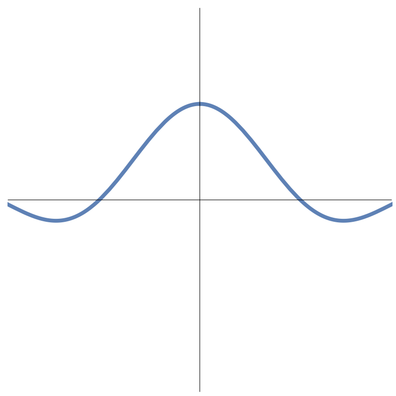

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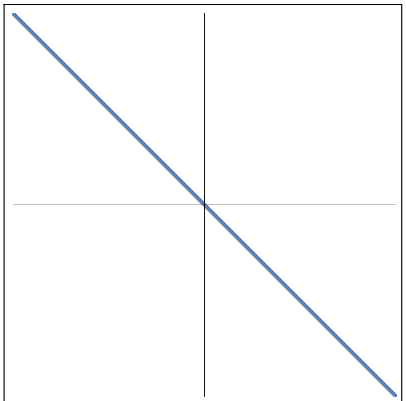

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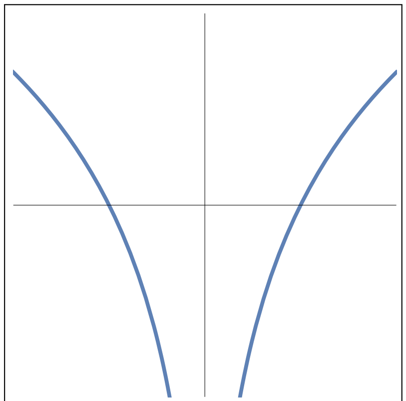

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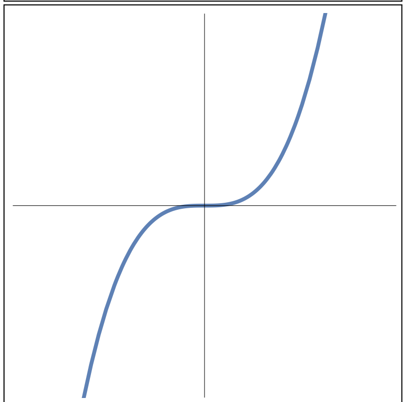

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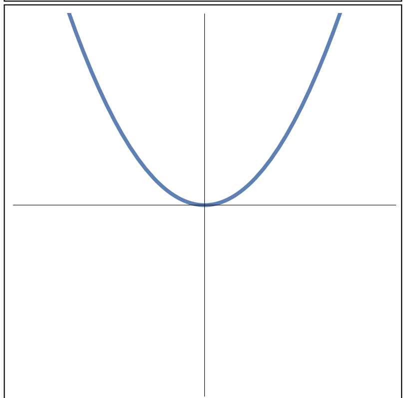

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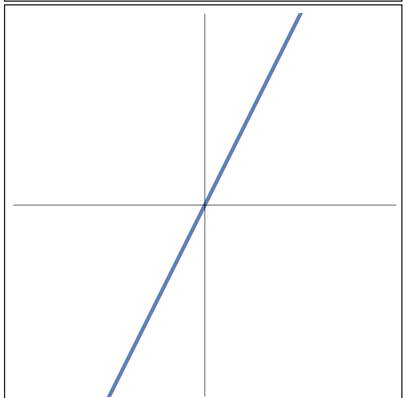
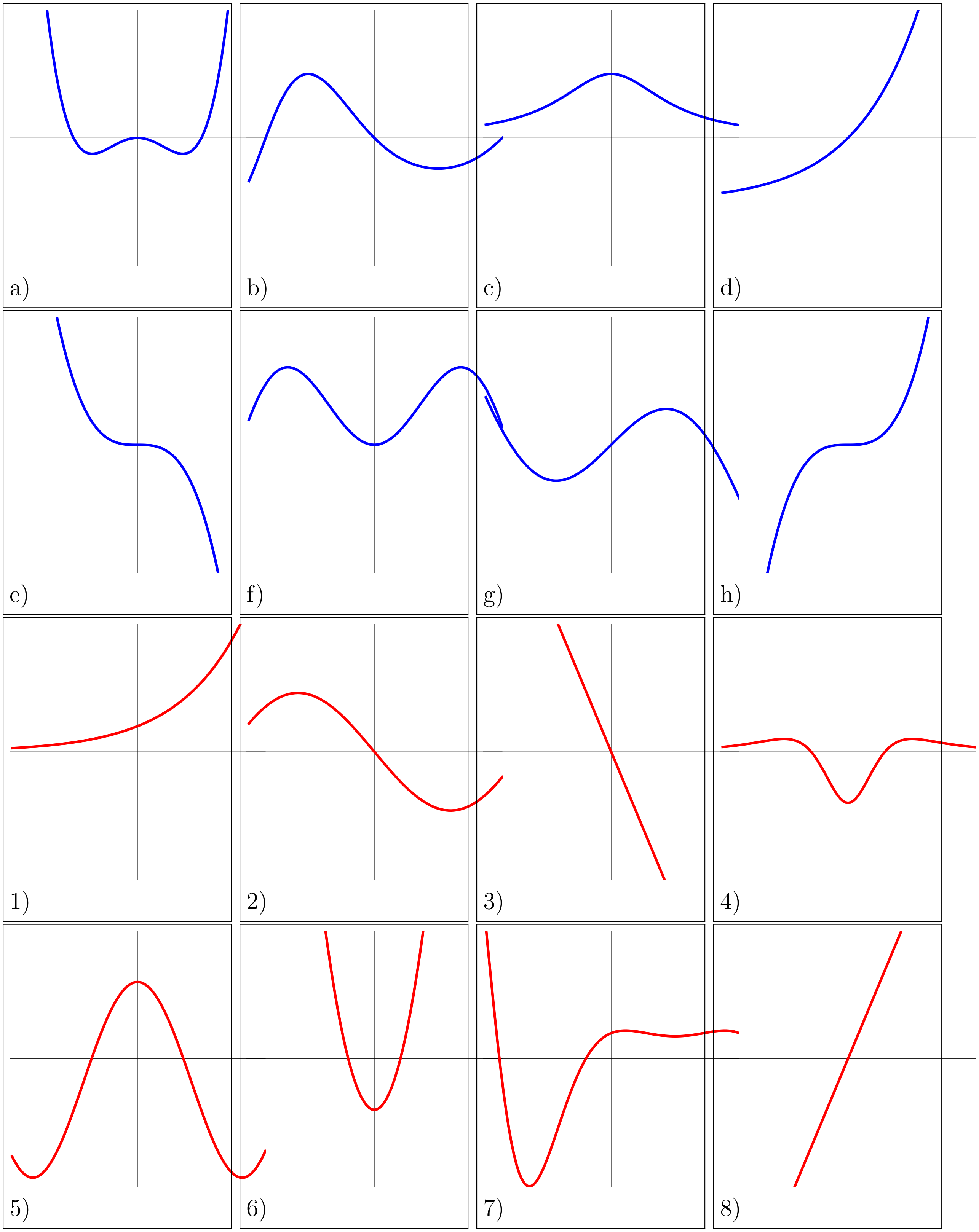


12)

Problem 3) Matching problem (10 points) No justifications are needed.

Match the graph of the functions f in $a) - h)$ with the second derivatives f'' in 1)-8).

Function	Second derivative (Enter 1- 8 here)
a)	
b)	
c)	
d)	
e)	
f)	
g)	
h)	



Problem 4) Continuity (10 points)

Some of the following functions might a priori not be defined yet at the point a. In each case, decide whether f can be made a continuous function by assigning a value $f(a)$ at the point a. If no such value exist, state that the function is not continuous.

a) (2 points) $f(x) = \frac{(x^3-1)}{(x-1)}$, at $x = 1$

b) (2 points) $f(x) = \sin(\frac{1}{x}) + \cos(x)$, at $x = 0$

c) (2 points) $f(x) = \sin(\frac{1}{\log(|x|)})$, at $x = 0$

d) (2 points) $f(x) = \log(|\sin(x)|)$, at $x = 0$

e) (2 points) $f(x) = \frac{(x-1)}{x}$, at $x = 0$

Problem 5) Chain rule (10 points)

a) (2 points) Write $1+\cot^2(x)$ as an expression which only involves the function $\sin(x)$.

b) (3 points) Find the derivative of the function $\text{arccot}(x)$ by using the chain rule for

$$\cot(\text{arccot}(x)) = x \ .$$

c) (2 points) Write $1+\tan^2(x)$ as an expression which only involves the function $\cos(x)$.

d) (3 points) Find the derivative of the function $\arctan(x)$ by using the chain rule for

$$\tan(\arctan(x)) = x \ .$$

Remark: even if you should know the derivatives of arccot or arctan, we want to see the derivations in b) and d).

Problem 6) Derivatives (10 points)

Find the derivatives of the following functions:

a) (2 points) $f(x) = \frac{\cos(3x)}{\cos(x)}$

b) (2 points) $f(x) = \sin^2(x)\log(1 + x^2)$

c) (2 points) $f(x) = 5x^4 - \frac{1}{x^2+1}$

d) (2 points) $f(x) = \tan(x) + \exp(-\sin(x^2))$

e) (2 points) $f(x) = \frac{x^3}{(1+x^2)}$

Problem 7) Limits (10 points)

Find the limits $\lim_{x \to 0} f(x)$ for the following functions f at $x = 0$ or state (providing reasoning as usual) that the limit does not exist.

a) (2 points) $f(x) = \frac{\sin(3x)}{\sin(x)}$

b) (2 points) $f(x) = \frac{\sin^2(x)}{x^2}$

c) (2 points) $f(x) = \sin(\log(|x|))$

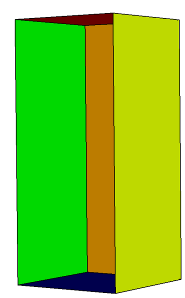
d) (2 points) $f(x) = \tan(x)\log(x)$

e) (2 points) $f(x) = \frac{(5x^4-1)}{(x^2+1)}$

Problem 8) Extrema (10 points)

A rectangular shoe-box of width x, length x and height y is of volume 2 so that $x^2y = 2$. The surface area adds up three rectangular parts of size $(x \times y)$ and 2 square parts of size $(x \times x)$ and leads to

$$f = 2x^2 + 3xy \ .$$



a) (2 points) Write down the function $f(x)$ of the single variable x you want to minimize.

b) (6 points) Find the value of x for which the surface area is minimal.

c) (2 points) Check with the second derivative test, whether the point you found is a local minimum.

Problem 9) Global extrema (10 points)

In this problem we study the function $f(x) = 3x^5 - 5x^3$ on the interval $[-2, 2]$.

a) (2 points) Find all roots of f.
b) (2 points) Find all local extrema of the function.
c) (2 points) Use the second derivative test to analyze the critical points, where applicable.
d) (2 points) Find the **global** maximum and minimum of f on the interval $[-2, 2]$.
e) (2 points) Bring the function $x\log(x)$ into indefinite form at 0.

Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

3/5/2021: First hourly Practice F

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False ⋯ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 6 AM on March 6th.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F 1 is the only root of the log function on the interval $(0,\infty)$.

2) T F $\exp(\log(5)) = 5$, if log is the natural log and $\exp(x) = e^x$ is the exponential function.

3) T F The function $\cos(x)+\sin(x)+x^2$ is continuous everywhere on the real axes.

4) T F The function $\sec(x) = 1/\cos(x)$ is the inverse of the function $\cos(x)$.

5) T F The function $x/\log(x)$ defines an indefinite form at ∞.

6) T F $\sin(3\pi/2) = -1$.

7) T F If a function f is continuous on $[0,\infty)$, then it has a global maximum on this interval.

8) T F The reciprocal rule assures that $d/dx(1/g(x)) = -1/g(x)^2$.

9) T F If $f(0) = g(0) = f'(0) = g'(0) = 0$ and $g''(0) = f''(0) = 1$, then $\lim_{x\to 0}(f(x)/g(x)) = 1$

10) T F An inflection point is a point, where the function $f''(x)$ changes sign.

11) T F If $f''(3) > 0$ then f is concave up at $x = 3$.

12) T F The intermediate value theorem assures that a continuous function has a maximum on a finite interval.

13) T F We can find a value b and define $f(1) = b$ such that the function $f(x) = (x^6-1)/(x^3-1)$ is continuous everywhere.

14) T F Single roots of the second derivative function f'' are inflection points.

15) T F If the second derivative $f''(x)$ is negative and $f'(x) = 0$ then f has a local maximum at x.

16) T F The function $f(x) = [x]^3 = x(x+h)(x+2h)$ satisfies $Df(x) = 3[x]^2 = 4x(x+h)$, where $Df(x) = [f(x+h)-f(x)]/h$.

17) T F The quotient rule is $d/dx(f/g) = (fg' - f'g)/g^2$.

18) T F The chain rule assures that $d/dx f(g(x)) = f'(g(x))f'(x)$.

19) T F With $Df(x) = f(x+1) - f(x)$, we have $D2^x = 2^x$.

20) T F Hospital's rule applied to the function $f(x) = \mathrm{sinc}(x) = \sin(x)/x$ gives us the fundamental theorem of trigonometry.

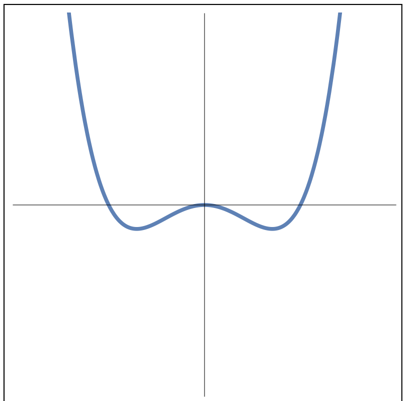
Problem 2) Matching problem (10 points) No justifications are needed.

Match the functions with the graphs.

Function	Enter 1-9
$1/(1+x^2)$	
$\cot(2x)$	
$3x+1$	

Function	Enter 1-9
$x\sin(x)$	
$\exp(-x)$	
$\log(\|x\|)$	

Function	Enter 1-9
$\text{sign}(x)$	
x^4-x^2	
$-x^2$	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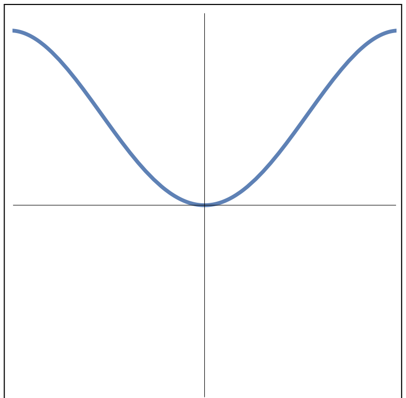
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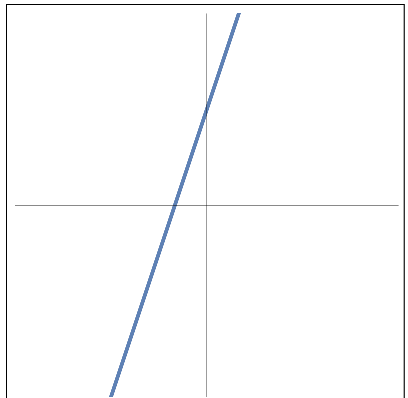
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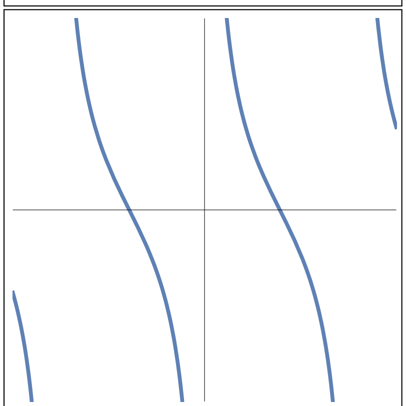
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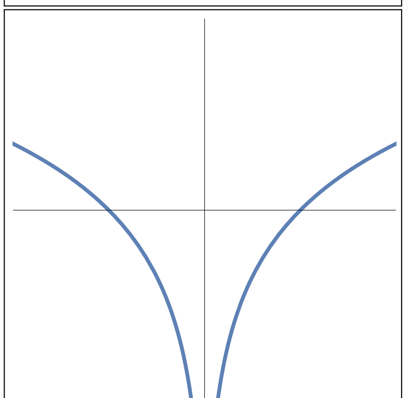
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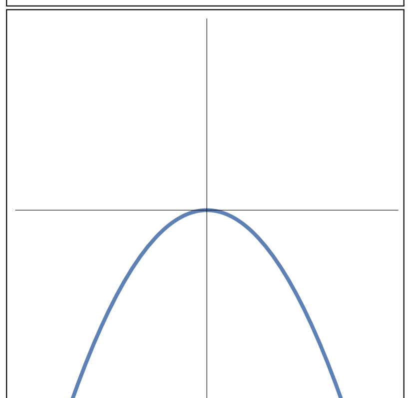
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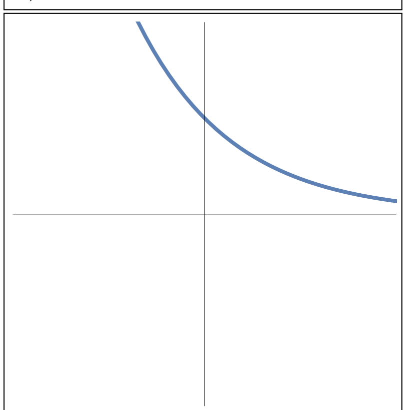
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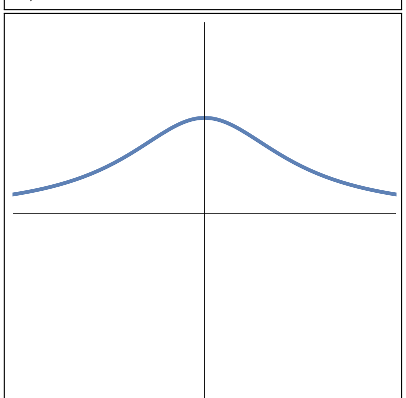
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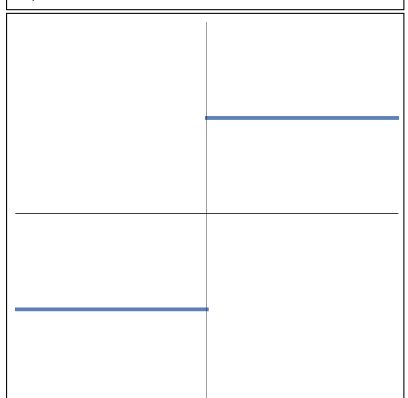
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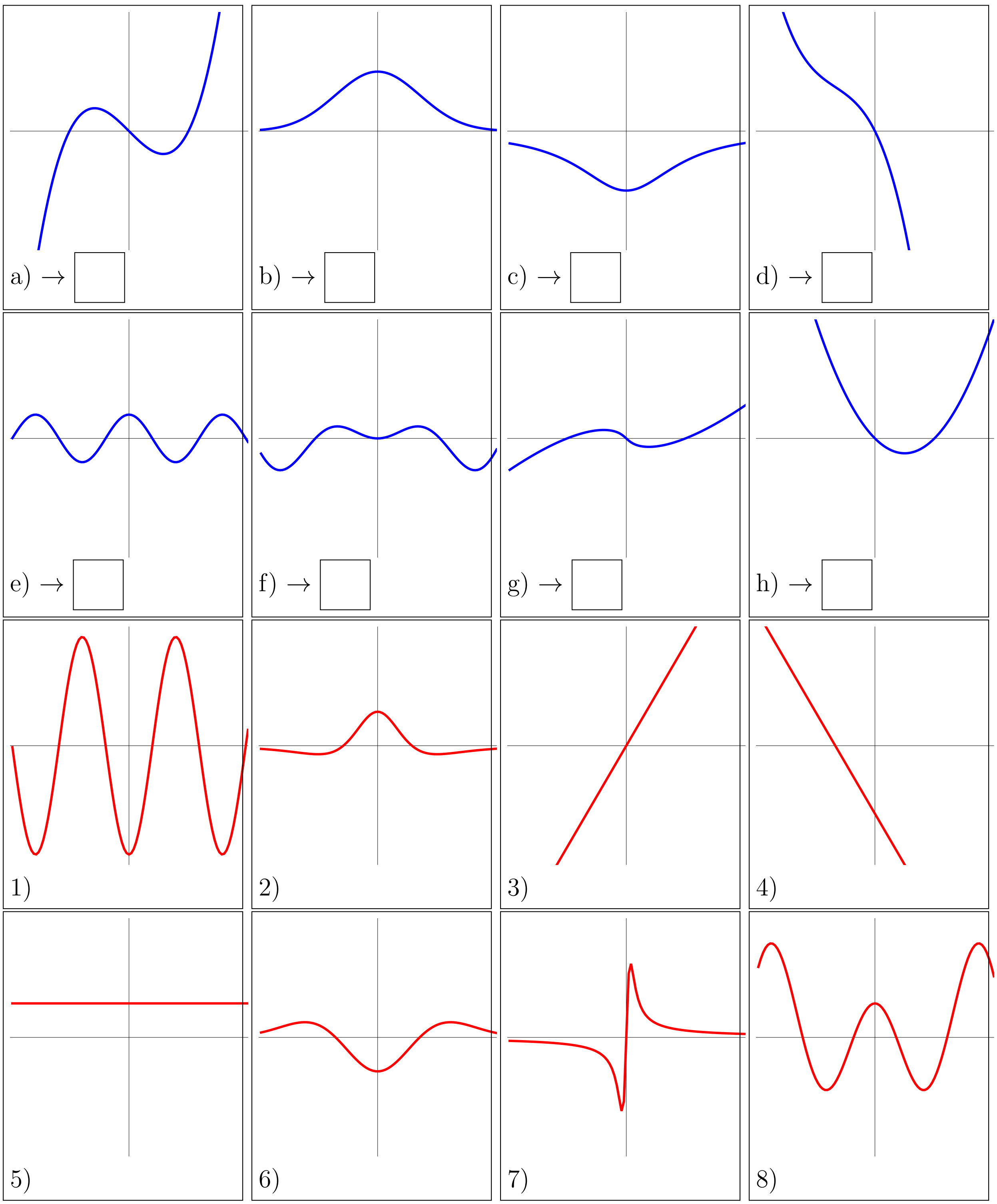
Problem 3) Matching problem (10 points) No justifications are needed.

Match the graph of the functions f in $a) - h)$ with the second derivatives f'' in 1)-8).



Problem 4) Continuity (10 points)

Decide whether the function can be healed at the given point in order to be continuous everywhere on the real line. If the function can be extended to a continuous function, give the value at the point.

a) (2 points) $f(x) = \frac{(x^3-8)}{(x-2)}$, at $x = 2$

b) (2 points) $f(x) = \sin(\sin(1/x)) - \tan(x)$, at $x = 0$

c) (2 points) $f(x) = \frac{\cos(x)-1}{x^2}$, at $x = 0$

d) (2 points) $f(x) = (\exp(x) - 1)/(\exp(5x) - 1)$, at $x = 0$

e) (2 points) $f(x) = \frac{(x-1)}{x}$, at $x = 0$

Problem 5) Chain rule (10 points)

In the following cases, we pretend not to know the formula for the derivative of log or arctan and again recover it using the chain rule.

b) (2 points) Rederive the derivative of the square root function $\text{sqrt}(x) = \sqrt{x}$ by differentiating

$$(\text{sqrt}(x))^2 = x$$

and solving for $\text{sqrt}'(x)$.

b) (4 points) Rederive the derivative of the logarithm function $\log(x)$ by differentiating

$$\exp(\log(x)) = x$$

and solving for $\log'(x)$.

c) (4 points) Rederive the formula for the derivative of the arctan function $\arctan(x)$ by differentiating the identity

$$\tan(\arctan(x)) = x$$

and using $1 + \tan^2(x) = 1/\cos^2(x)$ to solve for $\arctan'(x)$.

Problem 6) Derivatives (10 points)

Find the derivatives of the following functions:

a) (2 points) $f(x) = \frac{5\sin(x^6)}{x}$ for $x > 0$

b) (2 points) $f(x) = \tan(x^2) + \cot(x^2)$ for $x > 0$

c) (2 points) $f(x) = \frac{1}{x} + \log(x^2)$ for $x > 0$

d) (2 points) $f(x) = x^6 + \sin(x^4)\log(x)$ for $x > 0$

e) (2 points) $f(x) = \log(\log(x))$ for $x > 1$

Problem 7) Limits (10 points)

Find the limits $\lim_{x\to 0} f(x)$ for the following functions f at $x = 0$ or state that the limit does not exist. State the tools you are using.

a) (2 points) $f(x) = x^2 + x + \sin(1 - \cos(x))$

b) (2 points) $f(x) = \frac{x^3}{\sin(x^3)}$

c) (2 points) $f(x) = x^3/\sin(x)^2$

d) (2 points) $f(x) = x^3 + \text{sign}(x)$

e) (2 points) $f(x) = \cos(x^4) + \cos(\frac{1}{x})$

Problem 8) Extrema (10 points)

In the following problem you can ignore the story if you like and proceed straight go to the question:

Story: a cone shaped lamp designed in 1995 by **Verner Panton** needs to have volume $\pi r^2 h = \pi$ to be safe. To minimize the surface area $A = \pi r\sqrt{h^2 + r^2}$, we minimize the square A^2 and so $\pi^2 r^2(h^2 + r^2)$. From the volume assumption, we get $r^2 = 1/h$ so that we have to minimize $(\pi^2/h)(h^2 + 1/h) = \pi^2 f(h)$.

Which height h minimizes the function

$$f(h) = h + \frac{1}{h^2} \ ?$$

Use the second derivative test to check that you have a minimum.

Problem 9) Global extrema (10 points)

An investment problem leads to the profit function

$$f(x) = x - 2x^2 + x^3 ,$$

where $x \in [0, 2]$. Find the local and global maxima and minima of f on this interval and use the second derivative test.

Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

4/9/2021: Second hourly

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Your Name:

- Solutions are submitted to knill@math.harvard.edu as a PDF, handwritten in one file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids or online tools or external information are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 10 AM on April 10th. Do not communicate with anybody related to the class during the exam period and with nobody at all about the exam.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1)	T	F	The function $f(x) = \frac{e^{-(x^2/2)}}{\sqrt{2\pi}}$ is a PDF called the normal distribution.
2)	T	F	The method of partial fractions is based on the chain rule.
3)	T	F	$\int \log(x)\, dx = \frac{1}{x} + C$.
4)	T	F	If $0 \leq f(x) \leq 1$, then $0 \leq \int_0^1 f(x)\, dx \leq 1$
5)	T	F	A PDF is the anti-derivative of the CDF.
6)	T	F	If $f(x) = 1$ everywhere and $\int_a^b f(x)\, dx = 0$ then $a = b$.
7)	T	F	If f is continuous, then $\int_a^b f(-x)\, dx = -\int_a^b f(x)\, dx$.
8)	T	F	If F is a cumulative distribution function and $a < b$, then $F(b) - F(a)$ is the probability that the data are in the interval $[a, b]$.
9)	T	F	The function $e^x \cos(x)$ needs to be integrated by partial fractions.
10)	T	F	The fundamental theorem of calculus implies $\int_a^b f''(x)\, dx = f'(b) - f'(a)$ if f'' is a continuous function.
11)	T	F	The improper integral $\int_0^1 \frac{1}{x^{3/2}}\, dx$ is finite.
12)	T	F	Much of catastrophe theory has been developed and propagated by René Thom.
13)	T	F	$\int_1^2 \log(5x) dx$ is one fifth of $\int_1^2 \log(x)\, dx$.
14)	T	F	A CDF is monotonically increasing in the sense $F(x) \leq F(y)$ if $x \leq y$.
15)	T	F	If $\lim_{x \to \infty} f(x)^2 = 0$, then $\int_1^\infty f(x)\, dx$ is finite.
16)	T	F	An integral is called improper if it does not have a finite value.
17)	T	F	The anti-derivative of $\arctan(x)$ is $1/(1 + x^2) + C$.
18)	T	F	Tic-Tac-Toe integration method allows to integrate functions like $x^{10} e^x$.
19)	T	F	Gabriel's trumpet has infinite volume and finite surface area.
20)	T	F	The function $f(x) = e^{-x}$ for positive x and $f(x) = 0$ else is the exponential distribution.

Problem 2) Theorems (10 points) No justifications needed.

Fill in the missing part into the empty box to make a true statement. Read the statements carefully.

---

a) $\frac{d}{dx} \int_x^1 f(t)\, dt = f(1) -$ [ ] by the **fundamental theorem of calculus**.

---

b) Applying the iteration $T(x) = x - f(x)/f'(x)$ is called the [ ].

---

c) A **probability distribution** is a piece-wise continuous function which satisfies $\int_{-\infty}^{\infty} f(x)\, dx = 1$ and [ ].

---

d) The **improper integral** $\int_0^1 \frac{1}{x^p}\, dx$ converges, if p is contained in the open interval [ ].

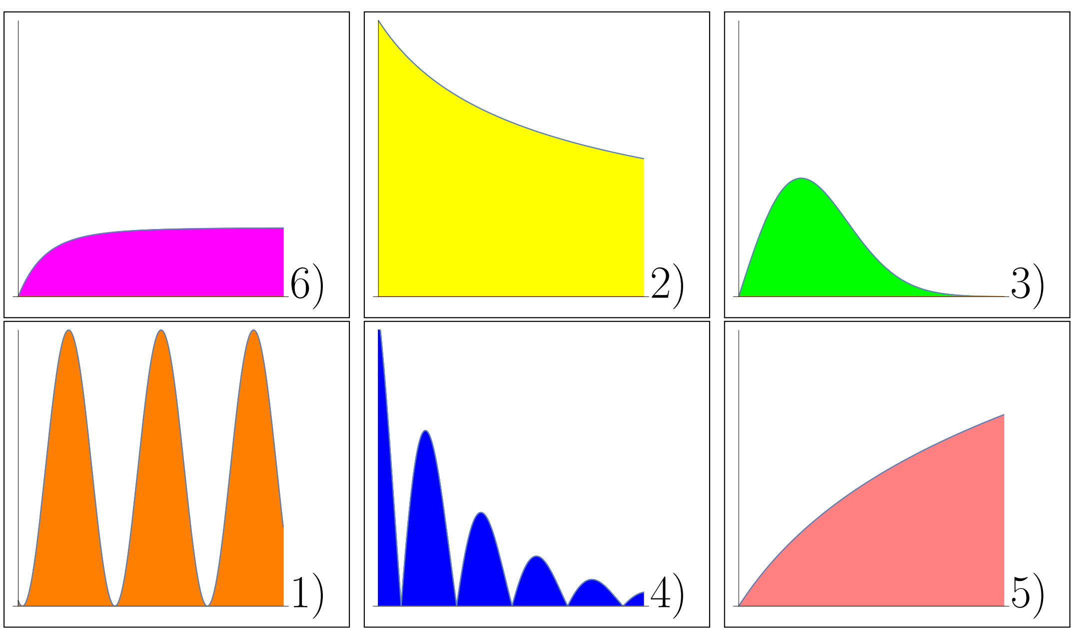
---

e) Assume $f_c(x)$ is a **family of functions** such that for $c < 0$, there are exactly 3 minima and for $c > 0$ there are exactly 2 minima, then c is called a [ ].

Problem 3) Matching problem (10 points)

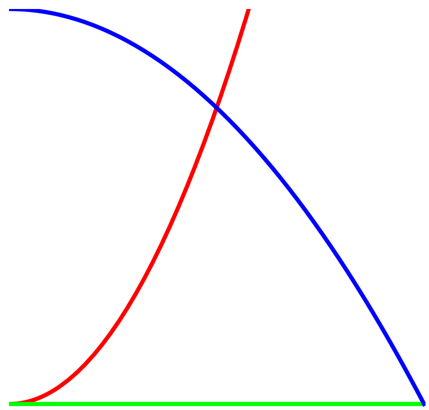
Match the following integrals with parts of the regions and indicate whether the integral is convergent or divergent.

Integral	1-6	Convergent	Divergent
$\int_1^\infty \sin^2(3x)\ dx$			
$\int_1^\infty \frac{1}{x^{1/2}}\ dx$			
$\int_1^\infty (x-1)e^{-(x-1)^2}\ dx$			
$\int_1^\infty \log(x)\ dx$			
$\int_1^\infty \frac{x^4-1}{x^4+1}\ dx$			
$\int_1^\infty 3\lvert\sin(5x)\rvert e^{-x}\ dx$			



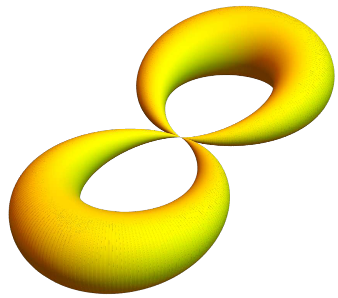
Problem 4) Area computation (10 points)

Find the **area of the triangular region** sandwiched between the graphs of $f(x) = 3x^2$ and $g(x) = 0$ and $h(x) = 4 - x^2$. Document your work.



Problem 5) Volume computation (10 points)

A solid has a cross section area of $A(x) = \pi \sin^2(x)$, where x goes from 0 to 2π. What is the **volume** V of that solid? As mentioned in class, the **Archimedes formula** still works also for the volume of the solid if it is bent around as long as the bending does not change the area of the cross section. The picture shows such a solid.



Problem 6) Definite integrals (10 points, 2 points each)

Compute the following definite integrals. State the integration steps and names.

a) $\int_0^1 2xe^{x^2+3}\,dx$
b) $\int_0^1 4x^3/(1+x^4)\,dx$
c) $\int_0^1 \frac{1}{(x-5)(x-3)}\,dx$
d) $\int_0^1 \log^2(1+x)\,dx$
e) $\int_0^1 xe^{1+x}\,dx$

Problem 7) Anti derivatives (10 points, 2 points each)

Solve the indefinite integrals. State the integration method in each case.

a) $\int \frac{\log(x)}{x^2}\,dx$
b) $\int \cos^2(x) - 3\sin^2(x)\,dx$
c) $\int \frac{1}{x^2-8x+12}\,dx$
d) $\int (x+1)^3 e^x\,dx$
e) $\int \frac{3x^2}{(1+x^6)}\,dx$

Problem 8) PDF's and CDF's (10 points)

We look at the function $f(x)$ which is $x^7/32$ for $0 \leq x \leq 2$ and $f(x) = 0$ else.

a) (4 points) Verify that $f(x)$ is a PDF.

b) (3 points) If data are distributed according to $f(x)$, what is the probability of the data to be in the interval $[a, b] = [0, 1]$?

c) (3 points) What is the mean of the distribution f?

Problem 9) Catastrophes (10 points)

Consider the family of functions $f(x) = x^3/3 + cx$ on the real line.

a) (4 points) Find all critical points of f for $c < 0$ and determine the stable ones or indicate there are none.

b) (4 points) Find all critical points of f for $c > 0$ and determine the stable ones or indicate there are none.

c) (2 points) For which value of c does a catastrophe occur?

4/9/2021: Second hourly, Practice A

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Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False ⋯ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 10 AM on April 10th. Do not communicate with anybody related to the class during the exam period and with nobody at all about the exam.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1)	T	F	The method of partial fractions allows to find anti-derivates of functions like $f(x) = 1/((x-77)(x+78))$
2)	T	F	The method of substitution is based on the chain rule.
3)	T	F	$\int x^3\,dx = 3x^2 + C$.
4)	T	F	The integral $\int_0^1 f(x)\,dx$ can be approximated by Riemann sums.
5)	T	F	If $f(x) = 1$ everywhere, then $\int_a^b f(x)\,dx$ is the length of the interval $[a, b]$.
6)	T	F	If f is continuous, then $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$.
7)	T	F	If f is a probability density function, then $f(b) - f(a)$ is the probability that the data are in the interval $[a, b]$.
8)	T	F	One can find the anti-derivative of $\sin(4x)\cos(17x)$ using integration by parts.
9)	T	F	The fundamental theorem of calculus implies $\int_a^b f'(x)\,dx = f(b) - f(a)$ if f' is a continuous function.
10)	T	F	If $f(x) = 1/\sqrt{x}$, then the improper integral $\int_0^1 f(x)\,dx$ exists and gives a positive finite area.
11)	T	F	The family $f_c(x) = x^2 + c$ experiences a catastrophe at $c = 0$.
12)	T	F	The anti derivative of $\log(5x)$ is $5x\log(5x) - 5x + C$.
13)	T	F	The volume of a cone of base radius 2 and height 2 is given by the integral $\int_0^2 \pi x^2\,dx$.
14)	T	F	The volume of a sphere can be derived from the volume of the cone and cylinder.
15)	T	F	If $\lim_{x\to\infty} f(x) = 0$, then $\int_1^\infty f(x)\,dx$ is finite.
16)	T	F	An integral is called improper if it is a convergent indefinite integral.
17)	T	F	The anti-derivative of $\cot(x)$ is $1/(1+x^2) + C$.
18)	T	F	Tic-Tac-Toe integration appears in the TV Series "Queen's Gambit".
19)	T	F	Gabriel's trumpet is a solid with infinite volume.
20)	T	F	The function $f(x) = e^x$ is called the exponential distribution.

21) [T] [F] A PDF is the anti-derivative of the CDF.

Problem 2) Theorems (10 points) No justifications needed.

Fill in the missing part into the empty box to make a true statement.

---

a) $\frac{d}{dx} \int_1^x f(t)\ dt =$ [ ] by the **fundamental theorem of calculus**.

---

b) $\int_1^x f'(t)\ dt =$ [ ] by the **fundamental theorem of calculus**.

---

c) A **probability distribution** is a piece-wise continuous function which is non-negative and satisfies the property [ ].

---

d) The **improper integral** $\int_1^\infty \frac{1}{x^p}\ dx$ converges, if p satisfies the property [ ].

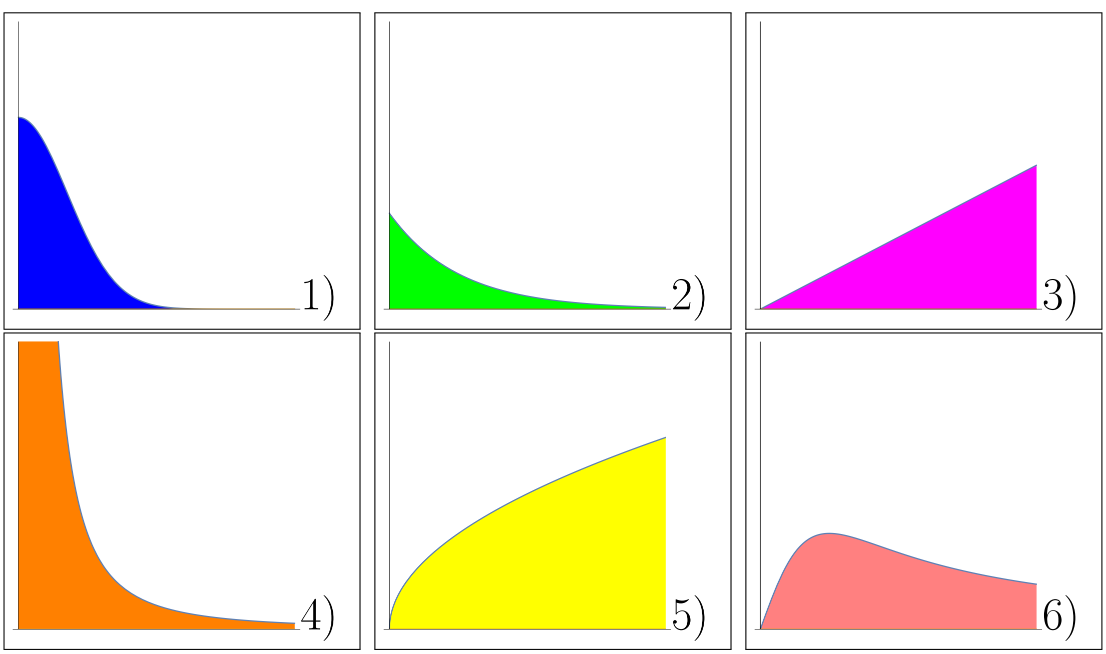
---

e) Assume $f_c(x)$ is a **family of functions** such that for $c < 0$, there is no minimum and for $c > 0$ there is one minimum, then c is called a [ ].

Problem 3) Matching problem (10 points)

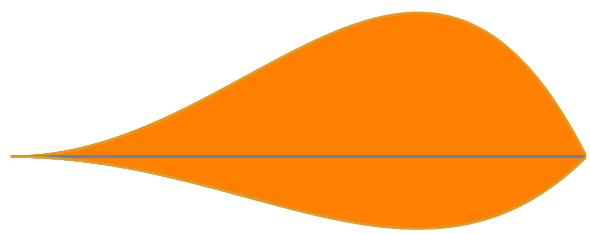
Match the following integrals with parts of the regions and indicate whether the integral represents a finite area.

Integral	Fill in 1-6	Convergent?
$\int_0^\infty x/2 \, dx$		
$\int_0^\infty \frac{1}{x^2} \, dx$		
$\int_0^\infty \sqrt{x} \, dx$		
$\int_0^\infty e^{-x} \, dx$		
$\int_0^\infty e^{-x^2} \, dx$		
$\int_0^\infty \frac{x}{1+x^2} \, dx$		



Problem 4) Area computation (10 points)

Find the **area of the region** sandwiched between the graphs of $f(x) = x^4 - x^2$ and $g(x) = 2x^2 - 2x^4$. Document your work.



Problem 5) Volume computation (10 points)

A solid has an elliptical cross section area of $A(x) = 1 - x^2$. It is located between $x = -1$ and $x = 1$. What is the **volume** V of that solid? Document your work.

Problem 6) Definite integrals (10 points, 2 points each)

Compute the following definite integrals. State the integration steps and names.

a) $\int_0^1 \frac{1}{(x+3)(x-4))} \, dx$

b) $\int_0^1 2x/(1+x^2) \, dx$

c) $\int_0^1 x^3/(1+x^2) \, dx$

d) $\int_0^1 \log(1+x) \, dx$

e) $\int_0^1 5xe^x \, dx$

Problem 7) Anti derivatives (10 points, 2 points each)

Solve the indefinite integrals. State the integration method in each case.

a) $\int x^2 \log(x) \, dx$

b) $\int \cos^2(x) \sin(x) \, dx$

c) $\int \cos^2(2x) \, dx$

d) $\int x^2 \sin(x) \, dx$

e) $\int 3x^2/(1+x^6) \, dx$

Problem 8) PDF's and CDF's (10 points)

We look at the function $f(x)$ which is $|x|$ on the interval $[-1, 1]$ and $f(x) = 0$ else.

a) (4 points) Verify that $f(x)$ is a PDF.

b) (3 points) If data are distributed according to $f(x)$, what is the probability of the data to be in the interval $[a, b] = [0, 1]$?

c) (3 points) What is the mean of the distribution f?

Problem 9) Catastrophes (10 points)

Let $f_c(x) = cx^2 + x^3$.

a) (3 points) Verify that $x = 0$ is a critical point for all c.

b) (4 points) Determine the stability of the critical point depending on c.

c) (3 points) For which value of c does a catastrophe appear?

Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

4/9/2021: Second hourly, Practice B

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Your Name:

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1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

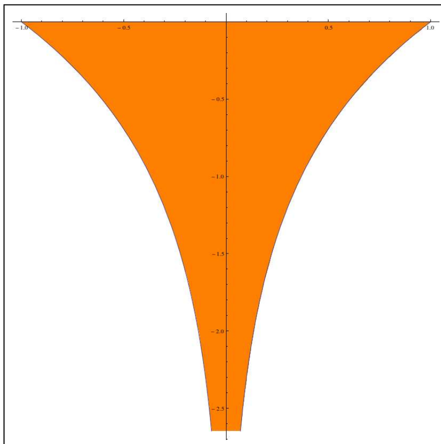
Problem 1) TF questions (20 points) No justifications are needed.

1)	T	F	The function $f(x) = \int x \sin(x)\, dx$ is integrated using the method of partial fractions.
2)	T	F	For any continuous function f we have $\int_0^1 3f(t)\, dt = 3\int_0^1 f(t)\, dt$.
3)	T	F	For any continuous function $\int_0^3 f(t)\, dt = 3\int_0^1 f(t)\, dt$.
4)	T	F	For any continuous function $\int_0^1 1 - f(t)\, dt = 1 - (\int_0^1 f(t)\, dt)$.
5)	T	F	The anti-derivative of $\tan(x)$ is $-\log(\cos(x)) + C$.
6)	T	F	The fundamental theorem of calculus implies that $\int_1^3 f'(x)\, dx = f(3) - f(1)$.
7)	T	F	The integral $\pi \int_0^1 x^2\, dx$ gives the volume of a cone of height 1.
8)	T	F	The anti-derivative of $1/\cos^2(x)$ is $\tan(x)$.
9)	T	F	The function $F(x) = \int_0^x \tan(t^2)\, dt$ has the derivative $\tan(x^2)$.
10)	T	F	The function $f(x) = \sin(x)/2$ on $[0, \pi]$ and $f(x) = 0$ else is a PDF
11)	T	F	The identity $\frac{d}{dx}\int_1^2 \log(x)\, dx = \log(2) - \log(1)$ holds.
12)	T	F	If $f < 1$, then $\int_0^2 f(x)\, dx$ can be bigger than 1.
13)	T	F	An improper integral is an improperly defined definite indefinite integral.
14)	T	F	The anti derivative $F(x)$ of $f(x)$ satisfies $F'(x) = f(x)$.
15)	T	F	A parameter value c for which the number of minima are different for parameters smaller or larger than c is called a catastrophe.
16)	T	F	If f is unbounded at 0, then $\int_0^1 f(x)\, dx$ is infinite.
17)	T	F	If $f(-1) = 0$ and $f(1) = 1$ then $f' = 2$ somewhere on $(-1, 1)$.
18)	T	F	The anti-derivative of $\log(x)$ is $x\log(x) - x + C$, where log is the natural log.
19)	T	F	The sum $\frac{1}{n}[(\frac{0}{n})^2 + (\frac{1}{n})^2 + ... + (\frac{n-1}{n})^2]$ converges to $1/3$ in the limit $n \to \infty$.
20)	T	F	The **improper integral** $\int_1^\infty \frac{1}{x^2}\, dx$ represents a finite area.

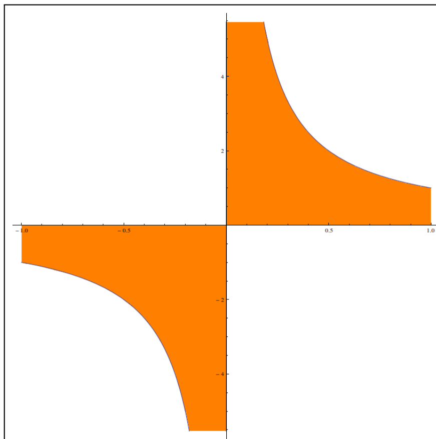
Problem 2) Matching problem (10 points) No justifications are needed.

a) (4 points) Match the following integrals with the regions and indicate whether the integral represents a finite area.

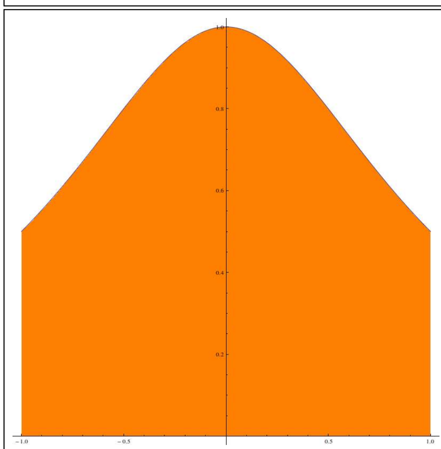
Integral	Fill in 1-4	Finite?
$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \, dx$		
$\int_{-1}^{1} \frac{1}{x} \, dx$		
$\int_{-1}^{1} \frac{1}{1+x^2} \, dx$		
$\int_{-1}^{1} \log\lvert x\rvert \, dx$		



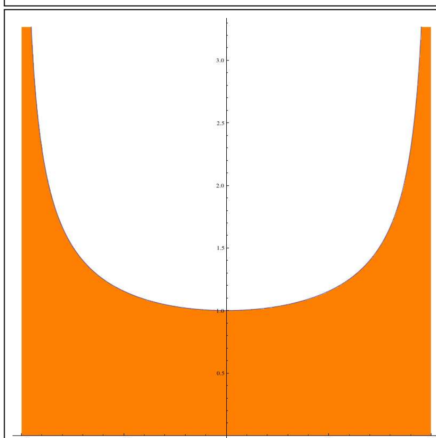
1)



2)



3)



4)

b) (6 points) Which of the following properties are always true. This means which are true for all choices of continuous functions and all choices of a, b, c.

Identity	Check if true
$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b f(x) + g(x)\,dx$	
$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$	
$\int_a^b f(x)^2\,dx = (\int_a^b f(x)\,dx)^2$	
$\int_a^a f(x)\,dx = 0$	
$\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$	

Problem 3) (10 points)

Fill in the missing part into the empty box to make a true statement:

a) (2 points)

$\frac{d}{dx}\int_0^x f(t)\,dt =$ [ ] by the **fundamental theorem of calculus**.

b) (2 points)

$\int_0^x f'(t)\,dt =$ [ ] by the **fundamental theorem of calculus**.

c) (2 points)

The **method of** [ ] writes $f(x) = 1/((x-6)(x+9))$ as $A/(x-6) + B/(x+9)$ and fixes the constants A, B.

d) (2 points)

A **probability distribution** satisfies $\int_{-\infty}^{\infty} f(x)\,dx = 1$ and [ ] for all x.

e) (2 points)

For an improper integral $\int_a^b f(x)\,dx$, either $a = \infty$ or $b = \infty$ or f is [ ] on $[a, b]$.

Problem 4) Area computation (10 points)

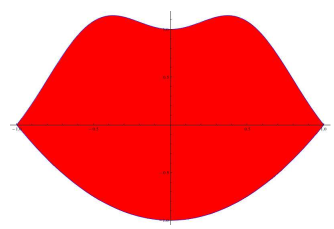
The region enclosed by the graphs of

$$f(x) = x^2 - 1$$

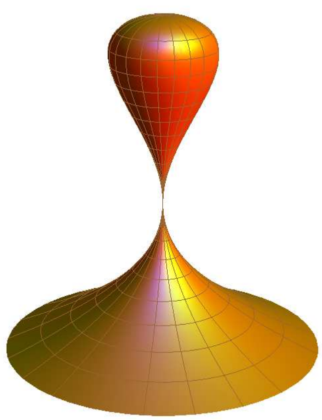
and

$$g(x) = 1 - x^2 + (1 - \cos(2\pi x))/6$$

models of the lips of **Rihanna**. Find the area.



Problem 5) Volume computation (10 points)



The **kiss** is a solid of revolution for which the radius at height z is

$$z^2\sqrt{1-z}$$

and where $-1 \leq z \leq 1$. What is the volume of this solid? The name "kiss" is the official name for this quartic surface. Indeed, the top part has the shape of a **Hershey Kiss**. P.S. Creative **"exam product placement"** like this has been invented and patented by Oliver himself ...

Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals. Each of the problems produces a numerical answer.

a) (2 points) $\int_0^1 \sqrt{1+x}\,dx$.

b) (2 points) $\int_1^2 \frac{1}{9-x^2}\,dx$

c) (2 points) $\int_2^e x \log(x) \, dx$

d) (2 points) $\int_1^2 \frac{x^2}{\sqrt{9-x^3}} \, dx$.

e) (2 points) $\int_0^\pi x^3 \sin(x) \, dx$

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives:

a) (2 points) $\int 23e^{23x} - x^{23} \, dx$

b) (2 points) $\int \frac{2}{x+3} + x^{1/23} \, dx$

c) (2 points) $\int \frac{23x}{1+x^2} + 23\tan(x) \, dx$

d) (2 points) $\int \log(x)^2 dx$

e) (2 points) $\int \cos^2(3x) \, dx$

Jim Carrey in the movie "The number 23"

Problem 8) PDF's and CDF's (10 points)

Verify that the function $f(x) = \exp(-|x|)/2$ is a PDF.

Problem 9) Catastrophes (10 points)

We look at the one-parameter family of functions $f_c(x) = 2x^3 + cx^2$, where c is a parameter.

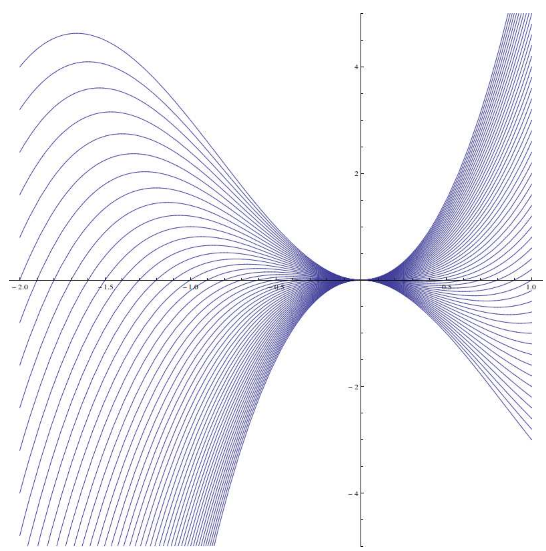
a) (2 points) Find the critical points of $f_3(x)$.

b) (2 points) Find the critical points of $f_{-3}(x)$.

c) (2 points) Check that 0 is always a critical point.

d) (2 points) For which c is 0 a minimum?

e) (2 points) For which c does the catastrophe occur?



4/9/2021: Second hourly, Practice C

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1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

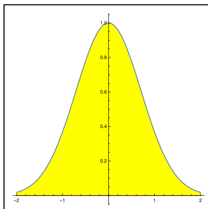
Problem 1) TF questions (20 points) No justifications are needed.

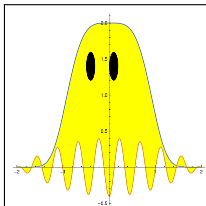
1)	T	F	If f is a continuous function then $\int_0^x f(t)\ dt$ is an area and therefore positive.
2)	T	F	The anti-derivative of $\text{arccot}(x)$ is $-\log(\sin(x)) + C$.
3)	T	F	The fundamental theorem of calculus implies that $\int_0^3 f''(x)\ dx = f'(3) - f(0)$.
4)	T	F	The volume of a cylinder of height 3 and radius 5 is given by the integral $\int_0^3 \pi 5^2\ dx$.
5)	T	F	The anti-derivative of $\tan(x)$ is $1/\cos^2(x)$.
6)	T	F	The mean value theorem implies that the derivative of $\sin(x)$ in the interval $[0, \pi/2]$ is $2/\pi$ somewhere.
7)	T	F	The function $F(x) = \int_0^x \sin(t^2)\ dt$ has the derivative $\sin(x^2)$.
8)	T	F	If f is a PDF, then $\int_{-\infty}^{\infty} xf(x)\ dx$ is called the mean.
9)	T	F	The identity $\frac{d}{dx} \int_0^1 \sin(x)\ dx = \sin(1)$ holds.
10)	T	F	If a solid is scaled by a factor 2 in all directions then its volume increases by a factor 8.
11)	T	F	If f is a PDF, then $\int_0^2 f(x)\ dx$ is the probability that the data are in $[0, 2]$.
12)	T	F	If $f(x)$ is smaller than $g(x)$ for all x, then $\int_0^1 f(x) - g(x)\ dx$ is negative.
13)	T	F	Every improper integral defines an infinite area.
14)	T	F	The anti derivative of $f'(x)$ is equal to $f(x) + c$.
15)	T	F	Catastrophes can explain why minima can change discontinuously.
16)	T	F	If f is discontinuous at 0, then $\int_{-1}^1 f(x)\ dx$ is infinite.
17)	T	F	If $f(-\infty) = 0$ and $f(\infty) = 1$ then $f' = 1$ somewhere on $(-\infty, \infty)$.
18)	T	F	The anti-derivative of $1/x$ is $\log(x) + C$, where log is the natural log.
19)	T	F	A catastrophe is defined as a critical point of f which is a minimum.
20)	T	F	The integral $\int_0^\infty 1/x^2\ dx$ represents a finite area.

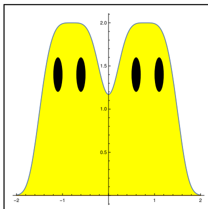
Problem 2) Matching problem (10 points) No justifications are needed.

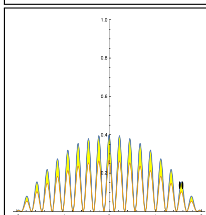
a) (6 points) Match the following integrals with the regions. Graphs 1) and 2) are inspired by a cartoon by Matthew Freeman (J Epidemiol. Community Health. 2006 January; 60(1): 6)

Integral	Fill in 1-4
$\int_{-2}^{2} (4-x^2)\cos^2(14x)/10 - (4-x^2)\cos(14x)/15 dx$	
$\int_{-2}^{2} 2\exp(-3(x+0.8)^4) + 2\exp(-3(x-0.8)^4\ dx$	
$\int_{-2}^{2} \exp(-x^2)\ dx$	
$\int_{-2}^{2} 2\exp(-x^4) - (x^2-4)\cos(14x)/10\ dx$	


1) Normal distribution


2) Paranormal distribution


3) Abnormal distribution


4) Wormal distribution

b) (4 points) The intermediate value theorem assures that $f'(x)$ takes the value $(f(a)-f(b))/(b-a)$ somewhere in the interval (a,b) for $a < b$. Which of the following statements follows from this?

Result	Check
If $f(0) = -1$ and $f(1) = 1$ then there is x with $0 \leq x \leq 1$ with $f'(x) = 2$	
If $f(0) = 1$ and $f(1) = 1$ then there is a critical point x of f in $(0,1)$	
If $f(0) = 1$ and $f(1) = 1$ then there is point where $f(x) = 2$ in $(0,1)$	
If $f(0) = 1$ and $f(1) = 1$ then there is point where $f''(p) = 0$ in $(0,1)$	

Problem 3) (10 points)

a) (4 points) Having seen some applications of integration and differentiation, complete the table: (the derivative of work is power, the anti derivative of marginal cost is cost.

Function f	Anti-derivative F
Probability density function	
	Total cost
	Mass
Area	
	Velocity
Power	
Velocity	

b) (2 points) We have seen two methods to find roots $f(x) = 0$ of equations. Both methods need some assumptions on the functions: Choose from the following: "differentiability", "continuity", "positivity".

Method	Assumption which f has to satisfy
Dissection method	
Newton method	

c) (2 points) Which is more general? Rolle's theorem assures that $f(a) = f(b)$, then $f'(x) = 0$ for some $a < x < b$. Related rates and implicit differentiation problems were not discussed this semester.

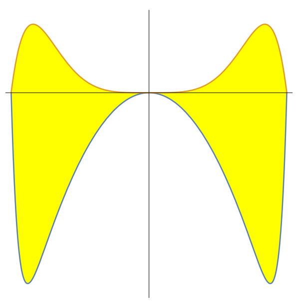
	Related rates	Implicit differentiation	
	Rolles theorem	Intermediate value theorem	

d) (2 points) Which integral is finite? Chose one!

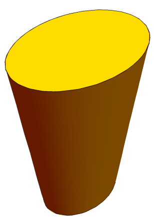
Integral	finite	infinite	
$\int_1^\infty 1/\sqrt{x}\, dx$			
$\int_1^\infty 1/x^2\, dx$			

Problem 4) Area computation (10 points)

The region enclosed by the graphs of $f(x) = x^{20} - x^2$ and $g(x) = x^4 - x^8$ is a cross section for a catamaran sailing boat. Find the area.

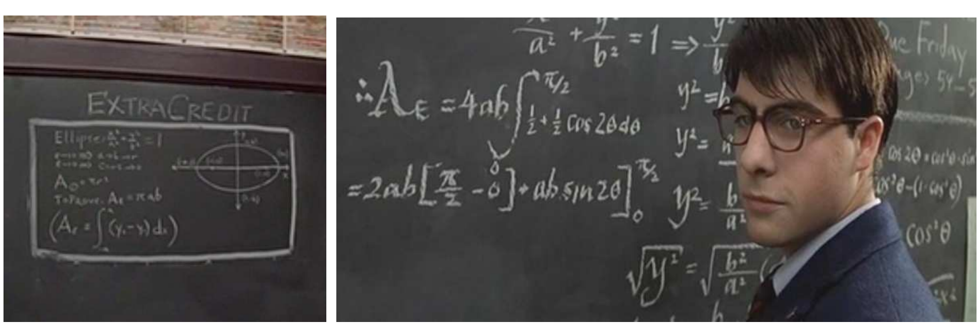


Problem 5) Volume computation (10 points)



We are given that an ellipse with diameters $2b$ and $2a$ has area πab. Find the volume of part of a cone whose height is between $z = 3$ and $z = 5$ for which the cross section at height z is an ellipse with parameters $a = 2z$ and $b = 3z$.

Remark. We will see later the area formula. In the movie **"Rushmore"**, the teacher tells about the problem: "I put that up as a joke. It's probably the hardest geometry equation in the world".



Screen shots from the movie Rushmore shows a blackboard where the formula for the ell.pdfe is computed using trig substitution. You might spot a double angle formula. We will come to that.

Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals. Each of the problems produces a numerical answer.

a) (2 points) $\int_0^1 (x-1)^4 \; dx$

b) (2 points) $\int_0^1 x \sin(\pi x) \; dx$.

c) (2 points) $\int_0^{\sqrt{3}} \frac{6}{1+x^2} \; dx + \int_0^{\sqrt{\pi}} \sin(x^2) 2x \; dx$

d) (2 points) $\int_1^2 \frac{1}{25-4x^2} \; dx$

e) (2 points) $\int_0^1 \frac{1}{\sqrt{1-x^2}} \; dx$.

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

a) (2 points) $\int e^{7x} - \log(x) \; dx$

b) (2 points) $\int \frac{5}{x+1} + 7\cos^2(x) \; dx$

c) (2 points) $\int \frac{11}{1+x^2} + 9\tan(x) \; dx$

d) (2 points) $\int \frac{4}{\cos^2(x)} + \frac{2}{\sin^2(x)} \; dx$

e) (2 points) $\int 2x \cos(x^2) \; dx$

Problem 8) PDF's and CDF's (10 points)

Verify that the function f which is defined to be $f(x) = 0$ for $x < 1$ and $f(x) = 3/x^4$ on $[1, \infty)$ is a PDF.

Problem 9) Catastrophes (10 points)

We look at the one-parameter family of functions $f_c(x) = x^6 - cx^4 - cx^2$, where c is a parameter.

a) (4 points) Verify that f has a critical point 0 for all c.

b) (3 points) Determine whether 0 is a minimum or maximum depending on c.

c) (3 points) For which c does a catastrophe occur?

Problem 10) Basic integrals (10 points)

Bonus problem: Find the anti derivatives. You have to solve in 10 seconds each. For every second over that limit, one point of the entire exam will be taken off. So, for example: if you use 62 seconds for the following 5 problems, you have used 12 seconds too much and 12 points are taken off from your exam. Don't worry, we do not assign negative points so that your final score will always remain a number between 0 and 110 points. But do not get too relaxed: if you choose not do this problem 10, 50 points are taken off.

a) (2 points) e^{-2x}.
b) (2 points) $\cos(15x)$.
c) (2 points) 2^x.
d) (2 points) $1/(1-x)$
e) (2 points) $1/(1+x^2)$

Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

4/9/2021: Second hourly, Practice D

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Your Name:

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- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 10 AM on April 10th. Do not communicate with anybody related to the class during the exam period and with nobody at all about the exam.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

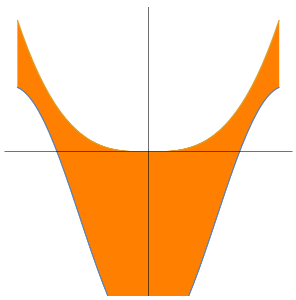
1)	T	F	The anti-derivative of $\tan(x)$ is $-\log(\cos(x)) + C$.
2)	T	F	The fundamental theorem of calculus implies that $\int_0^1 f'(x)\,dx = f(1) - f(0)$.
3)	T	F	The volume of truncated pyramid with a base square length 2 and top square length 3 is given by the integral $\int_2^3 x^2\,dx$.
4)	T	F	The derivative of $\arctan(x)$ is $1/\cos^2(x)$.
5)	T	F	The mean value theorem implies $\int_a^b f'(x)\,dx = f'(c)(b-a)$ for some c in the interval (a,b).
6)	T	F	If $F(x) = \int_0^x f(t)\,dt$ has an critical point at $x = 1$ then f has a root at $x = 1$.
7)	T	F	The anti-derivative of the derivative of f is equal to $f + C$ where C is a constant.
8)	T	F	The CDF is an anti-derivative of a PDF.
9)	T	F	The identity $\frac{d}{dx}\int_5^9 f(x)\,dx = f(9) - f(5)$ holds for all continuous functions f.
10)	T	F	Two surfaces of revolution which have the same cross section area $A(x)$ also have the same volume.
11)	T	F	The integral $\int_{-\infty}^0 e^x\,dx$ is an improper integral which converges.
12)	T	F	The identity $\int_2^9 7f(x)\,dx = 7\int_2^9 f(x)\,dx$ is true for all continuous functions f.
13)	T	F	The improper integral $\int_1^\infty 1/x\,dx$ in the sense that $\int_1^R 1/x\,dx$ converges for $R \to \infty$ to a finite value.
14)	T	F	If $f_c(x)$ has a local minimum at $x = 2$ for $c < 1$ and no local minimum anywhere for $c > 1$, then $c = 1$ is a catastrophe.
15)	T	F	An improper integral is an indefinite integral which does not converge.
16)	T	F	If $f(-5) = 0$ and $f(5) = 10$ then $f' = 1$ somewhere on the interval $[-5, 5]$.
17)	T	F	The sum $\frac{1}{n}\sum_{k=0}^{n-1}\frac{k}{n} = \frac{1}{n}[\frac{0}{n} + \frac{1}{n} + \cdots + \frac{n-1}{n}]$ is a Riemann sum to the integral $\int_0^1 x\,dx$.
18)	T	F	The anti-derivative of $\mathrm{sinc}(x) = \sin(x)/x$ is equal to $\sin(\log(x)) + C$.
19)	T	F	The anti-derivative of $\log(x)$ is $1/x + C$.
20)	T	F	We have $\int_0^x t f(t)\,dt = x\int_0^x f(t)\,dt$ for all functions f.

Problem 2) Matching problem (10 points) No justifications are needed.

a) (6 points) Match the integrals with the pictures.

Integral	Enter 1-6
$\int_{-1}^{1}(1-x)^2\,dx$	
$\int_{-1}^{1}\lvert x\rvert\,dx$	
$\int_{-1}^{1}x^4\,dx$	

Integral	Enter 1-6
$\int_{-1}^{1}\lvert x\rvert^3-\cos(3x)\,dx$	
$\int_{-1}^{1}[\sin^2(\pi x)-\cos^2(\pi x)]\,dx$	
$\int_{-1}^{1}1-\lvert x\rvert\,dx$	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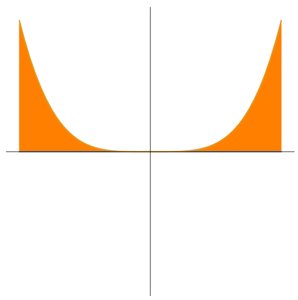
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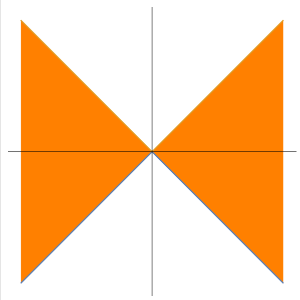
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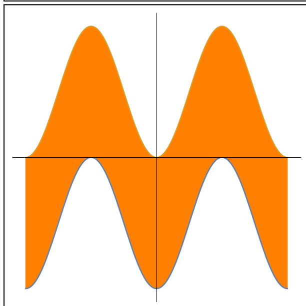
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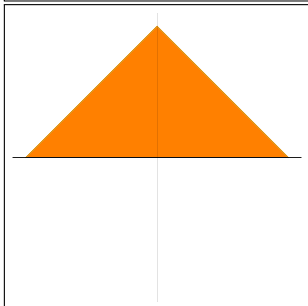
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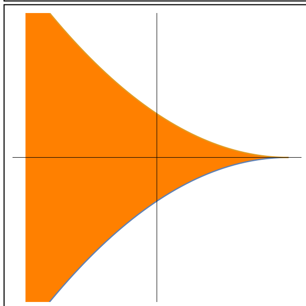
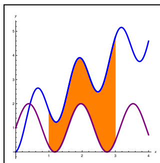

6)

b) (4 points) Match the concepts: each of the 4 figures illustrates one of the formulas which are the centers of the **mind map** we have drawn for this exam: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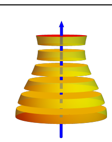
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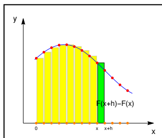
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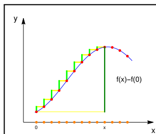

4)

Formula	Enter 1-4
$\int_a^b A(z)\,dz$	
$\int_a^b g(x)-f(x)\,dx$	

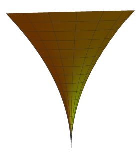
Formula	Enter 1-4
$\frac{d}{dx}\int_0^x f(t)\,dt=f(x)$	
$\int_0^x f'(t)\,dt=f(x)-f(0)$	

Problem 3) Matching problem (10 points) No justifications are needed.

a) (6 points) Match the volumes of solids.

Integral	Enter 1-6
$\int_0^1 \pi z^4 \, dz$	
$\int_0^1 \pi z \, dz$	
$\int_0^1 \pi(4+\sin(4z)) \, dz$	

Integral	Enter 1-6
$\int_{-1}^1 \pi e^{-4z^2} \, dz$	
$\int_0^1 \pi z^2 \, dz$	
$\int_0^1 (1-z)^2 \, dz$	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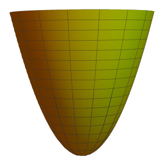
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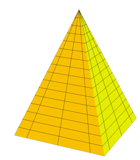
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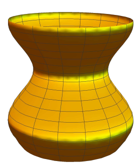
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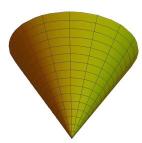
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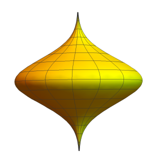

6)

b) (4 points) Fill in the missing word which links **applications** of integration.

The probability density function is the		of the cumulative distribution function.
The total cost is the		of the marginal cost.
The volume of a solid is the		of the cross section area function.
The velocity of a ball is the		of the acceleration of the ball.

Problem 4) Area computation (10 points)

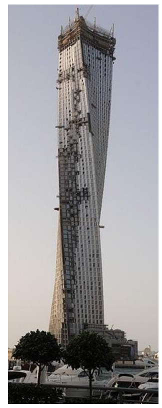
Find the area of the region enclosed the graphs of $y = x^4 - 12$ and $y = 8 - x^2$.

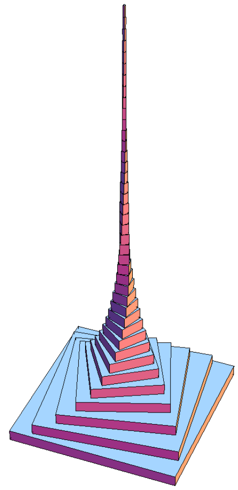
Problem 5) Volume computation (10 points)

The **infinity tower** in Dubai of height 330 meters has floors which can rotate. After much delay, it is expected to be completed this year. Inspired by the name "infinity", we build a new but twisted science center for which the side length of the square floor is

$$l(z) = \frac{1}{1+z} \; .$$

Find the volume of this new **Harvard needle building** which extends from 0 to ∞. We are the best!





Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals. You should get a definite real number in each case.

a) (2 points) $\int_0^\infty xe^{-x} \, dx$

b) (3 points) $\int_0^1 5x^{1/5} + 3x^3 + 3\sqrt{1+x^2}x \, dx$.

c) (3 points) $\int_{-1}^1 \frac{1}{1+x^2} \, dx$

d) (2 points) $\int_0^{e-1} \frac{2}{1+x} \, dx$

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

a) (2 points) $\int \frac{3}{\sqrt{1+3x}} + \cos^2(x) \, dx$

b) (3 points) $\int e^{x/5} - 7x^6 + \frac{4x}{x^2+1} \, dx$

c) (2 points) $\int \frac{4}{e^{4x+5}} + 3(x-1)\sin(x) \, dx$

d) (3 points) $\int \frac{1}{\sin^2(x)} + \frac{4}{x^2-1} \, dx$

Problem 8) PDF's and CDF's (10 points)

In order to verify that if f is a PDF and g is a PDF, then $(f+g)/2$ is a PDF, which three conditions do we have to check?

a) The function is piecewise [ ]

b) The function is [ ].

c) The integral [ ] is equal to [ ].

Problem 9) Catastrophes (10 points)

Verify first for each of the following functions that $x = 0$ is a critical point. Then give a criterium for stability of $x = 0$. The answer will depend on c.

a) (3 points) $f(x) = x^5 + 2x^2 - cx^2$.
b) (3 points) $f(x) = x^4 + cx^2 - x^2$.

Determine now in both examples for which parameter c the catastrophe occurs

c) (2 points) in the case $f(x) = x^5 + 2x^2 - cx^2$.
d) (2 points) in the case $f(x) = x^4 + cx^2 - x^2$.

Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

4/9/2021: Second hourly, Practice E

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- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 10 AM on April 10th. Do not communicate with anybody related to the class during the exam period and with nobody at all about the exam.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1)	T	F	The formula $\int_0^x f''(x)\,dx = f'(x) - f'(0)$ holds.
2)	T	F	The area of the upper half disc is the integral $\int_{-1}^{1} \sqrt{1-x^2}\,dx$
3)	T	F	If the graph of the function $f(x) = x^2$ is rotated around the interval $[0,1]$ in the x axes we obtain a solid with volume $\int_0^1 \pi x^4\,dx$.
4)	T	F	The function $f(x) = e^x$ is the only anti derivative of e^x.
5)	T	F	If f has a critical point at 1, then $F(x) = \int_0^x f(t)\,dt$ has an inflection point at 1.
6)	T	F	Catastrophes are parameter values c for a family of functions $f_c(x)$, for which a local minimum of f_c disappears.
7)	T	F	The volume of a cylinder of height and radius 1 minus the volume of a cone of height and radius 1 is half the volume of a sphere of radius 1.
8)	T	F	The function $f(x) = 1/(\pi\sqrt{1-x^2})$ for $-1 < x < 1$ and $f(x) = 0$ else is a PDF
9)	T	F	The improper integral $\int_0^1 x^{1/3}\,dx$ is finite.
10)	T	F	Integrals are linear: $\int_0^x f(t) + g(t)\,dt = \int_0^x f(t)\,dt + \int_0^x g(t)\,dt$.
11)	T	F	The function $\mathrm{Li}(x) = \int_2^x dt/\log(t)$ has an anti-derivative which is a finite construct of trig functions.
12)	T	F	There is a region enclosed by the graphs of x^5 and x^6 which is finite and positive.
13)	T	F	The integral $\int_{-1}^{1} 1/x^4\,dx = -1/(5x^5)\|_{-1}^{1} = -1/5 - 1/5 = -2/5$ is defined and negative.
14)	T	F	Gabriel's trumpet has finite volume but infinite surface area.
15)	T	F	A function $f(x)$ is a probability density function, if $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$.
16)	T	F	The mean of a function on an interval $[a,b]$ is $\int_a^b f(x)\,dx$.
17)	T	F	The cumulative probability density function is an anti-derivative of the probability density function.
18)	T	F	The integral $\int_{-\infty}^{\infty} (x^2 - 1)\,dx$ is finite.
19)	T	F	The total prize is the derivative of the marginal prize.
20)	T	F	The acceleration is the anti-derivative of the velocity.

Problem 2) Matching problem (10 points) No justifications are needed.

Match the following functions with their anti derivatives. Of course only 6 of the 30 functions will appear.

Function	Anti-derivative Enter 1-30
$\cos(3x)$	
$\sin(3x)$	
$3x$	

Function	Anti-derivative Enter 1-30
$1/(3x)$	
$\tan(3x)$	
$1/(1+9x^2)$	

1) $\sin(3x)$
2) $-\sin(3x)/3$
3) $\sin(3x)/3$
4) $-3\sin(3x)$
5) $3\sin(3x)$

6) $\cos(3x)$
7) $-\cos(3x)/3$
8) $\cos(3x)/3$
9) $-3\cos(3x)$
10) $3\cos(3x)$

11) $\log(x)/3$
12) $1/(3-x)$
13) $1/(3x)$
14) $\log(x/3)$
15) $-1/(3x^2)$

16) $3x^2$
17) $x^2/2$
18) $3x^2/2$
19) 3
20) x^2

21) $\arctan(3x)/3$
22) $3\arctan(3x)$
23) $1/(1+9x^2)$
24) $3/(1+9x^2)$
25) $-3/(1+x^2)$

26) $1/\cos^2(3x)$
27) $\log(\cos(3x))$
28) $-\log(\cos(3x))/3$
29) $\log(\cos(3x))/3$
30) $3/\cos^3(3x)$

Problem 3) Matching problem (10 points) No justifications are needed.

Which of the following formulations is a Riemann sum approximating the integral $\int_0^3 f(x)\,dx$ of $f(x)=x^2$ over the interval $0,3$.

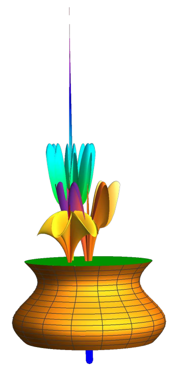
Sum	Check if this is the Riemann sum.
$n\sum_{k=0}^{n-1}(3k/n)^2$	
$\frac{1}{n}\sum_{k=0}^{n-1}(3k/n)^2$	
$n\sum_{k=0}^{3n-1}(k/n)^2$	
$\frac{1}{n}\sum_{k=0}^{3n-1}(k/n)^2$	

Problem 4) Area computation (10 points)

Find the area of the region enclosed by the three curves $y=6-x^2, y=-x$ and $y=x$ which is above the x axes.

Problem 5) Volume computation (10 points)

Emma Woodhouse grows plants in a pot which is a rotationally symmetric solid for which the radius at position x is $5 + \sin(x)$ and $0 \leq x \leq 2\pi$. Find the volume of the pot.



Problem 6) Definite integrals (10 points)

Find the following definite integrals

a) (5 points) $\int_1^2 x^2 \sin(\pi x)\, dx$.

b) (5 points) $\int_1^3 x^2 \cos(x^3 + 2)\, dx$

Problem 7) Anti-derivatives (10 points)

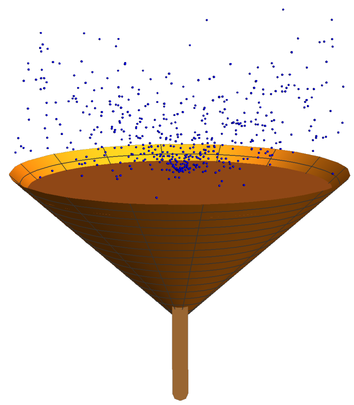
Find the following anti-derivatives

a) (5 points) $\int \frac{3}{\sqrt{1-x^2}} + x^4 + \frac{1}{1+x^2}\, dx$

b) (5 points) $\int \frac{1}{(x-2)(x+4)} + \frac{2}{x-1}\, dx$

Problem 8) Chain rule and volume (10 points)

A coffee machine has a filter which is a cone of radius z at height z. If the coffee covers everything until height t, let $V(t)$ denote the volume and $z(t)$ the height at time t. Coffee spills out at a rate of $V'(t) = 1$ cubic centimeter per second. How fast does the water level sink at height $z = 10$? To solve this, note that the chain rule equates $d/dtV(t)$ with $\pi z^2 z'(t)$.



Problem 9) PDF's and CDF's (10 points)

Assume we know that the CDF is given by $f(x) = 1/2 + \arctan(x)/\pi$.
Determine the corresponding PDF.

Problem 10) Improper integrals (10 points)

Evaluate the following improper integrals or state that they do not exist

a) (3 points) $\int_1^\infty 1/\sqrt{x}\, dx$.

b) (2 points) $\int_0^1 \sqrt{x}\, dx$.

c) (3 points) $\int_0^\infty 2xe^{-x^2}\, dx$.

d) (2 points) $\int_0^\infty \frac{1}{x}\, dx$

Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

4/9/2021: Second hourly, Practice F

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- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 10 AM on April 10th. Do not communicate with anybody related to the class during the exam period and with nobody at all about the exam.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F The formula $\int_0^x f''(x)\,dx = f'(x) - f'(0)$ holds.

2) T F The area of the lower half disc is the integral $\int_{-1}^{1} -\sqrt{1-x^2}\,dx$

3) T F If the graph of the function $f(x) = x^2$ is rotated around the interval $[0,1]$ we obtain a solid with volume $\int_0^1 \pi x^4\,dx$.

4) T F The identity $d/dx \int_0^x f''(t)\,dt = f'(x)$ holds.

5) T F There is a point in $[0,1]$, where $f'(x) = 0$ if $f(x) = x^3 - x^2 + 1$.

6) T F The fundamental theorem of calculus assures that $\int_a^b f'(x)\,dx = f(b) - f(a)$.

7) T F If f is differentiable on $[a,b]$, then $\int_a^b f(x)\,dx$ exists.

8) T F The integral $\int_0^{\pi/2} \sin(\sin(x))\,dx$ is positive.

9) T F The anti-derivative of an anti-derivative of f is equal to the derivative of f.

10) T F If a function is positive everywhere, then $\int_a^b f(x)\,dx$ is positive too.

11) T F If a differentiable function is odd, then $\int_{-1}^{1} f(x)\,dx = 0$.

12) T F If $f_c(x)$ is a function with a local minimum at 0 for all $c < 0$ and no local minimum in $[-1,1]$ for $c > 0$, then $c = 0$ is called a catastrophe.

13) T F The term "improper integral" is a synonym for "indefinite integral".

14) T F The function $F(x) = x\sin(x)$ is an anti-derivative of $\sin(x)$.

15) T F The mean value theorem holds for every continuous function.

16) T F Newton and Leibniz were best buddies all their life. Leibniz even gave once the following famous speech: "You guys might not know this, but I consider myself a bit of a loner. I tend to think of myself as a one-man wolf pack. But when my sister brought Isaac home, I knew he was one of my own. And my wolf pack ... it grew by one.

17) T F Any function $f(x)$ satisfying $f(x) > 0$ is a probability density function.

18) T F The moment of inertia integral I can be used to compute energy with the relation $E = \omega^2 I/2$ where ω is the angular velocity.

19) T F If $0 \le f(x) \le g(x)$ then $0 \le \int_0^1 f(x)\,dx \le \int_0^1 g(x)\,dx$.

20) T F The improper integral $\int_0^{\infty} 1/(x^4+1)\,dx$ is finite.

Problem 2) Matching problem (10 points) No justifications are needed.

From the following functions there are two for which no elementary integral is found. Find them. You can find them by spotting the complement set of functions which you can integrate.

Function	Anti-derivative is not elementary	Function	Anti-derivative is not elementary
e^{-x^2}		$1/\log(x)$	
$\sin(3x)$		$\tan(3x)$	
$1/x$		$\arctan(3x)$	

Problem 3) Matching problem (10 points) No justifications are needed.

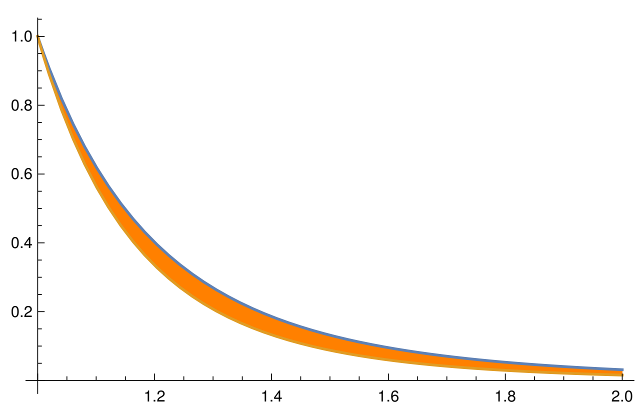
Which of the following functions are PDF's, which are CDF's, which are neither?

Function	PDF	CDF
$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$		
$\mathrm{Erf}(x)$		
$f(x) = e^{-\lvert x\rvert}/2$		
$f(x) = \frac{1}{\pi} \frac{1}{1+x^2}$		
$f(x) = \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}}$		

Problem 4) Area computation (10 points)

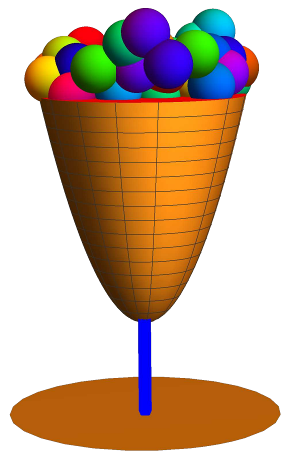
a) (5 points) Find the area of the region enclosed by the curves $3 - x^4$ and $3x^2 - 1$.

b) (5 points) Find the area of the region between $1/x^6$ and $1/x^5$ from $x = 1$ to $x = \infty$.



Problem 5) Volume computation (10 points)

Cody eats some magic "Bertie Botts Every Flavor Beans" from a cup which is a rotationally symmetric solid, for which the radius at position x is $\sqrt{x}$ and $0 \leq x \leq 4$. Find the volume of Cody's candy cup.



Problem 6) Definite integrals (10 points)

Find the following definite integrals

a) (4 points) $\int_1^3 (x+1)^3 + 1 \, dx$

b) (6 points) $\int_1^2 2x \tan(x^2) + \sin(x) + \cos(x) + \log(x) \, dx$.

Problem 7) Anti derivatives (10 points)

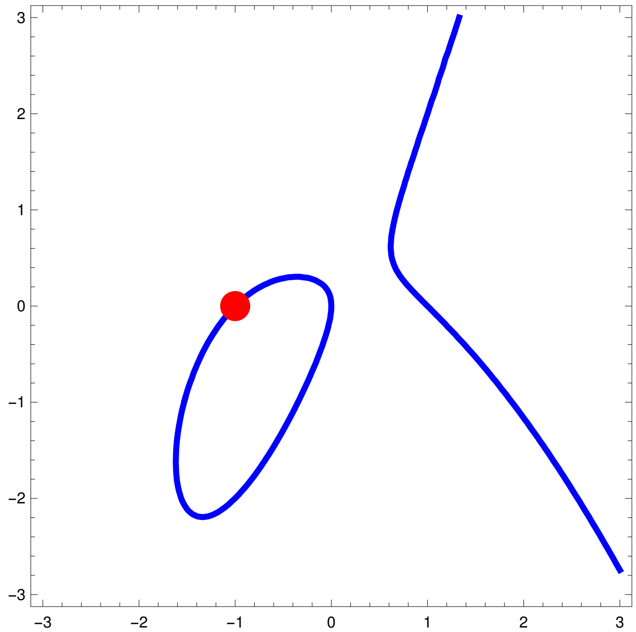
Find the following anti-derivatives

a) (5 points) $\int x^2 \sin(x^3) \, dx$

b) (5 points) $\int x e^{x-4} + (3/2)/\sqrt{x^5} \, dx$

Problem 8) Chain rule (10 points)

The curve $y^2 = x^3 + 2xy - x$ is an example of an **elliptic curve**. Use the chain rule to compute dy/dx at the point $(-1, 0)$ without solving for y first. Just differentiate both sides $2yy' = 3x^2 + 2xy' + 2y - 1$.



Problem 9) Applications (10 points)

The probability density of the exponential distribution is given by $f(x) = (1/2)e^{-x/2}$. The probability to wait for for time x (hours) to get an idea for a good calculus exam problem is $\int_0^x f(x)\, dx$. What is the probability to get a good idea if we wait for $T = 10$ (hours)?

Problem 10) Applications (10 points)

What is the **average value** of the function

$$f(x) = 4 + 1/(1 + x^2)$$

on the interval $[-1, 1]$? The average value is defined as $\int_a^b x f(x)\, dx/(b - a)$.

5/7/2021: Final Exam

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- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 9 AM on May 8th.

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6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points). No justifications are needed.

1) T F If f is a PDF, then $\int_{-\infty}^{\infty} x^2 f(x)\, dx$ is called the variance of the distribution.

2) T F The function $\log(e^x)$ is defined and continuous for all real numbers x.

3) T F There exists $p > 0$ such that $\int_0^\infty x^p\, dx$ exists and is a finite value.

4) T F $(e^x)^y = (e^y)^x$ for all real numbers x, y.

5) T F $e^{(x^y)} = e^{(y^x)}$ for all real numbers x, y.

6) T F The tangent function is continuous everywhere.

7) T F There was a time in you life when your age $t \in \mathbb{R}$ in units of years was exactly your height $h \in \mathbb{R}$ in units in meters.

8) T F The limit $\lim_{x\to 0} x\log(|x|)$ is 0.

9) T F The Monte-Carlo method is a numerical integration method.

10) T F One of the double angle formulas is $\sin(2x) = \frac{(1-\cos^2(x))}{2}$.

11) T F The function $f(x) = x(1 - x^5) + \sin(\pi x)$ has a critical point in $(0, 1)$.

12) T F The function $f(x) = \sin(x)/x$ is called the entropy function.

13) T F If $Df(x) = f(x+1) - f(x)$ and $Sf(x) = f(0) + f(1) + \cdots + f(x-1)$, then $DSf(x) = f(x)$ and $SDf(x) = f(x) - f(0)$.

14) T F The function $1/\log(2 - |x|)$ is defined and continuous for all real numbers x.

15) T F The Newton iteration method allows to find the roots for any continuous function.

16) T F The logarithm function $\log(x)$ is monotonically increasing for all $x > 0$.

17) T F Applying the Newton step with the function $f(x) = x^2 - 2$ at the point $x = 1$ gives $T(x) = 1 + 1/2 = 3/2$.

18) T F The family of functions $f_c(x) = c(x-1)^2 + 3$ experiences a catastrophe at $c = 0$.

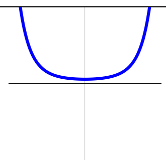
19) T F If $F(x) = x^2 + 3x$ then $f(x) = 2x + 3$ is called marginal cost.

20) T F The gradus suavitatis of the fraction 3/8 is equal to 6.

Problem 2) Matching problem (10 points) No justifications needed

(5 points) Match the name of the functions with their graphs (1-4), with their derivatives (A-D) (middle row) and with the second derivatives (a-d) (last row). If you write on your own paper, please copy the table first.

Function	fill in 1)-4)	fill in A)-D)	fill in a)-d)
$1/(1+x^2)$			
$\sin(x^3)$			
$x^3 - x$			
$\frac{e^{1+x^2}}{20}$			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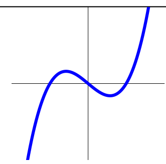
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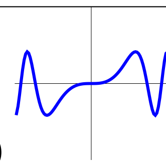
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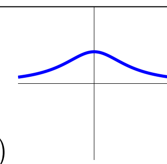
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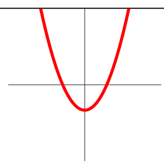
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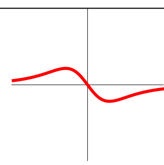
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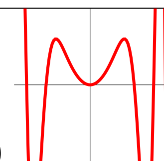
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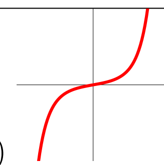
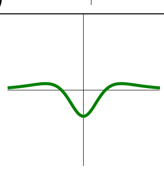
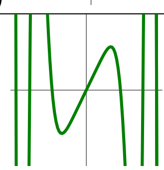

D)


a)

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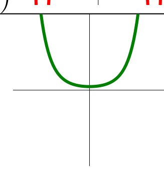
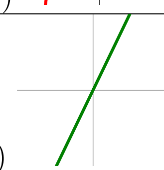

c)

d)

(5 points) In which cases can we take the limit $x \to 0$? If there is a limit, enter it in the left column, otherwise cross check the right column. If you write on your own paper, please copy the table first.

Function	The limit is (if it exists)	Cross check if not existing
$\frac{\sin(17x)}{\sin(23x)}$		
$-x \log \lvert 3x \rvert$		
$\frac{\sin(x^2)}{\sin^2(x)}$		
$\log \lvert 5x \rvert / \log \lvert 7x \rvert$		
$\arctan(x)/\tan(x)$		
$\frac{\cos(x)+1}{x^2}$		

Problem 3) Short answer problem (10 points). No justifications are needed.

a) (3 points) Complete the following table of probability distributions and cumulative distribution functions.

PDF	PDF supported on	CDF on that interval
e^{-x}	$[0, \infty)$	
	$(-\pi/2, \pi/2)$	$\frac{\arcsin(x)}{\pi} + \frac{1}{2}$
	$(-\infty, \infty)$	$\frac{\arctan(x)}{\pi} + \frac{1}{2}$

b) (3 points) We integrate $\int_0^1 x\, dx$ numerically, using only $n = 1$ interval and compare three different integration methods. What values do we get in each case?

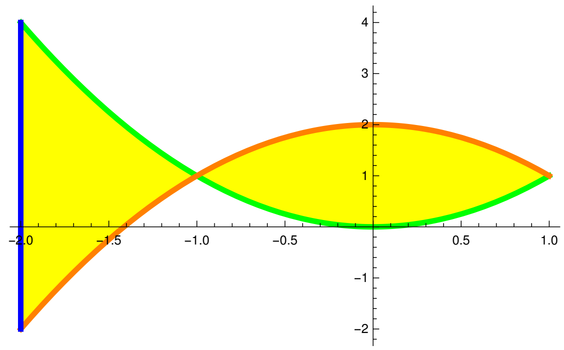
Integration method	Which value does the method give?
Trapezoid method	
Simpson Method	
Simpson 3/8 Method	

c) (2 points) If f is the marginal cost and F the total cost and g the average cost. What is the definition of the **break even point** in this context?

d) (2 points) What theorem is responsible for the fact that there is a point on earth such that the temperature on P and its anti-pod point Q are exactly the same?

Problem 4) Area computation (10 points)

What is the area of the **fish shape** region bordered by the curves $y = x^2, y = 2-x^2, x = -2$? We stress that we want the **area** of the region and **not** the signed area of the region.

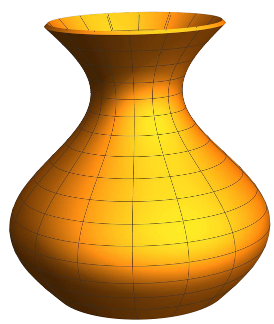


Problem 5) Volume computation (10 points)

Compute the volume of the **vase** which has at height z a **circular cross sections of radius**

$$r(z) = 2 + \sin(z)$$

and where the height z is in the interval $0 \leq z \leq 2\pi$.



Problem 6) Improper integrals (10 points)

You see here five improper integrals. Pick the ones which converge and compute their value. For the others, state that the integral does not exist and give a short explanation.

a) (2 points) $\int_0^1 x^{-6} \, dx$

b) (2 points) $\int_0^1 x^{-1/6} \, dx$

c) (2 points) $\int_1^\infty x^{-6}\ dx$

d) (2 points) $\int_1^\infty x^{-1/6}\ dx$

e) (2 points) $\int_0^\infty x^{-6}\ dx$.

Problem 7) Extrema (10 points)

A **trapezoid** of height $2x$, bottom length x and top length $(4 - 2x^2)$ has area

$$f(x) = x(x + (4 - 2x^2)) = 4x + x^2 - 2x^3 \ .$$

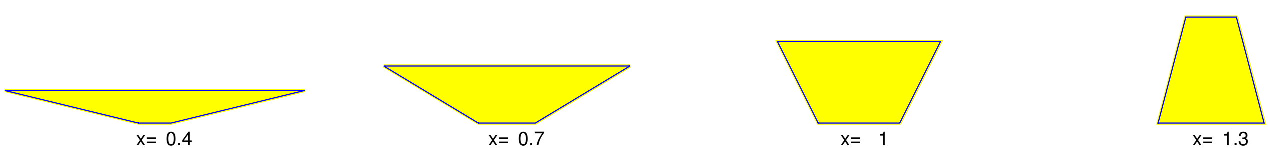
The figure below illustrates the situation for some x values.

a) (3 points) For which x does f have a local maximum? Use the second derivative test to check.

b) (3 points) On which interval $[a, b]$ does the problem make sense (height, lengths and area can not be negative)? What are the function values on the boundary?

c) (2 points) There a theorem which assures that $f(x)$ has a global maximum on that interval $[a, b]$? What is the name of the theorem?

d) (2 points) What is the global minimum of $f(x)$ on the interval of consideration?



Problem 8) Integration by parts (10 points)

a) (5 points) Compute the following anti-derivative:

$$\int (x - 2)^4 e^{x/2}\ dx \ .$$

b) (5 points) And now have some fun riding the "merry go round" for the following integral

$$\int e^x \sin(x) \; dx \; .$$

Problem 9) Substitution/chain rule (10 points)

a) (5 points)

$$\int \cos(\cos(\cos(x))) \sin(\cos(x)) \sin(x) \; dx \; .$$

b) (5 points)

$$\int \frac{2\log(\log(\log(x)))}{\log(\log(x))\log(x)x} dx \; .$$

Problem 10) Advanced integration (10 points)

a) (5 points) Integrate

$$\int \frac{1}{(x+8)(x+4)(x+2)} \; dx \; .$$

b) (5 points) Use the magic **trig substitution box** to find the anti-derivative:

$$\int \frac{1}{\cos(x)} + \frac{1}{\sin(x)} dx \; .$$

Here is the magic box:

$u = \tan(x/2)$
$dx = \frac{2du}{(1+u^2)}$
$\sin(x) = \frac{2u}{1+u^2}$
$\cos(x) = \frac{1-u^2}{1+u^2}$

Problem 11) PDF's and CDF's. (10 points)

We look at two functions which are defined to be zero outside the interval $[0, \pi]$ and which are given on $0 \leq x \leq \pi$ as

$$f(x) = \frac{\sin(x)}{2}$$

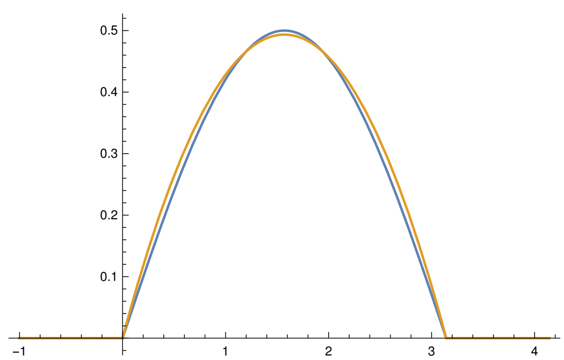
and

$$g(x) = \frac{x(\pi - x)}{5} .$$

Their graphs are very close to each other, as you can see in the picture.

a) (5 points) One of these functions produces a probability distribution. Which one? Justify your answer.

b) (5 points) The other function needs to be multiplied with a constant c to become a probability distribution. What is this constant?



Problem 12) Which integration method? (10 points)

a) (5 points) Determine the value of the definite integral

$$\int_0^{\pi/2} e^{\sin(x)} \cos(x) \, dx .$$

b) (5 points) Find the anti-derivative

$$\int 3\sqrt{1-x^2} + \frac{4}{\sqrt{1-x^2}}\, dx \ .$$

Problem 13) Applications (10 points) Only answers are needed

a) (2 points) What is the expectation $\int x f(x)\, dx$ of the probability distribution function $f(x) = e^{-|x|}/2$ which is a function defined for all real x. [We only need a number.]

b) (2 points) What is the inverse function of the (fictional) **Stockhausen Midi function** $f(x) = 440 \cdot 5^{(x-25)/5}$? This function gives the midi number x as a function of the frequency f.

c) (2 points) Which digit among $0, 1, \dots, 8, 9$ is the **least frequent** in **Benford's distribution law**? [You only need to provide the digit]

d) (2 points) The family of functions $f_c(x) = x^4 - cx^2 + x^2$ experiences a **catastrophe**. What is the parameter c for which this happens? [You only need to find that parameter.]

e) (2 points) If $(-2, -7)$ and $(2, 7)$ are two data points. What is the best linear fit $y = ax + b$ which passes through the two points?

Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

5/7/2021: Final Practice A

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13		10
Total:		140

Problem 1) TF questions (20 points). No justifications are needed.

1) T F $\sin(3\pi/2) = -1$

2) T F The cotangent function is monotonically decreasing on the open interval $(\pi/4, \pi/2)$.

3) T F The arccot function is monotonically increasing from 1 to 2.

4) T F If F is a CDF, then $F(0) = 0$

5) T F $\frac{d}{dx} \log(e^x) = 1$, where as always log is the natural log.

6) T F The limit of $\sqrt{|x|}/\sin(\sqrt{|x|})$ for $x \to 0$ exists and is equal to 1.

7) T F If we apply the l'Hospital rule for the limit $\lim_{h\to 0}[f(x+h) - f(x)]/h$ we get $f'(x)$

8) T F If $f'(1) = 0$ and $f''(0) = 1$ then f has a local minimum at $x = 0$.

9) T F The improper integral $\int_{-1}^{1} 1/\sqrt{1-x^2}\, dx$ is finite.

10) T F The function $f(x) = 1 + \sin(x^2) - x^3$ has a root in the interval $(-100, 100)$.

11) T F If a function f has a local minimum and a local maximum, then it must have a second minimum.

12) T F To any continuous function f, there exists a unique $F(x)$ such that $F'(x) = f(x)$.

13) T F If you listen to the sound $\log(1+x)\sin(10000x)$, then it gets louder and louder as time goes on.

14) T F The function $f(x) = e^{-x^2}$ has a local minimum at $x = 0$

15) T F The function $f(x) = (x^{25} - 1)/(x^5 - 1)$ has the limit 20 for $x \to 1$.

16) T F If the average cost $F(x)/x$ of an entity is extremal at $x = 2$, then we have a break-even point $f(2) = g(2)$.

17) T F If f is a PDF, then $\int_{-\infty}^{\infty} x^2 f(x)\, dx$ is called the variance of f.

18) T F The Midi function $f(s)$ gives the midi number $f(s)$ as a function of the frequency s.

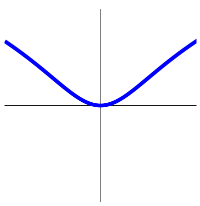
19) T F A Newton step for the function f is $T(x) = x - \frac{f(x)}{f'(x)}$.

20) T F $\sin(\arcsin(1)) = 1$.

Problem 2) Matching problem (10 points) No justifications needed

(5 points) Match the functions names with their graphs (1-4) their derivatives (A-D) (middle row) and second derivatives (a-d) (last row).

Function	fill in 1)-4)	fill in A)-D)	fill in a)-d)
$\sin(x)$			
$\cos(x^2)$			
$\log(1+x^2)$			
$\exp(-x^2)$			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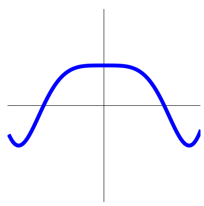
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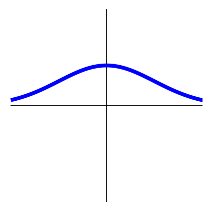
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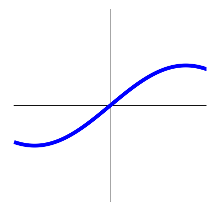
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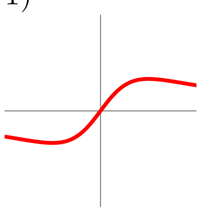
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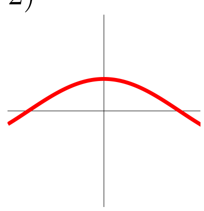
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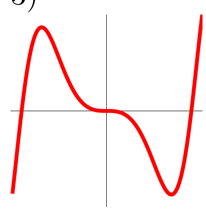
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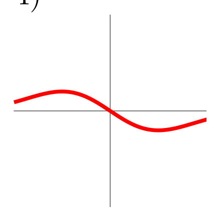
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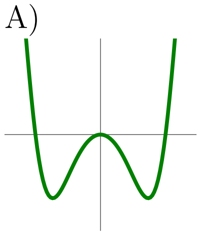
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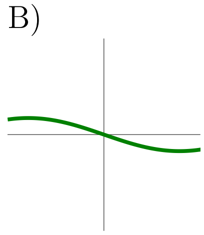
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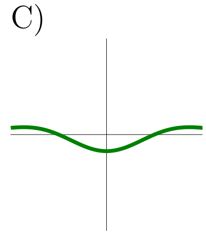
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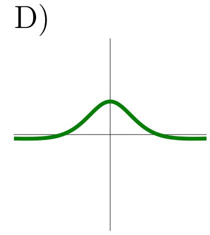

d)

(5 points) Which of the following limits exist in the limit $x \to 1$? If the limit exists, enter the result

Function	Enter the limit if it exists	Check if it does not exist
$\frac{(1-x^9)}{(1-x^7)}$		
$\frac{x}{\log\lvert x\rvert}$		
$\frac{\tan(1-x)}{(1-x)}$		
$\log\lvert x\rvert/x$		
$\log(x)/\log(2x)$		
$\frac{x^2-1}{\sin(x^2-1)}$		

Problem 3) Short answer problem (10 points). No justifications are needed.

a) (3 points) Find the relation between the following functions:

function f	function g	$f = g'$	$g = f'$	none
$\log(x)$	$1/x$			
$1/x$	$-1/x^2$			
$\tan(x)$	$1/(1+x^2)$			
$\cot(x)$	$-1/\sin^2(x)$			
$\arctan(x)$	$1/\cos^2(x)$			
$\text{arccot}(x)$	$-1/(1+x^2)$			

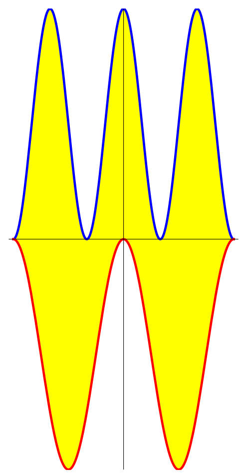
b) (3 points) We integrate $\int_0^1 f(x)\, dx$ numerically. During the numerical integration method lecture we have pointed out that some integration methods give always exact answers for quadratic functions f. For which numerical integration methods is this the case?

Integration method	The method gives the exact value for quadratic f
Archimedes (equal spacing)	
General Riemann sum	
Trapezoid method	
Simpson Method	
Simpson 3/8 Method	

c) (2 points) Formulate the "Strawberry theorem" in economics.

d) (2 points) Which mathematical theorem is involved for the "wobbly table theorem"?

Problem 4) Area computation (10 points)



Find the area enclosed by the graphs of the functions

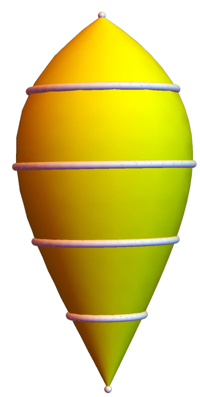
$$f(x) = \cos(2\pi x) - 1 \; .$$

and

$$g(x) = 1 + \cos(3\pi x)$$

on an interval on which f has two minima and g has three maxima. The situation is displayed in the picture.

Problem 5) Volume computation (10 points)



As kids we used to play with a **wooden top**, which is brought in motion by pulling at a rope wound around the solid. We want to the volume of such a top which at height z is a disk of radius

$$r(z) = z(1 - z^3)$$

and where the z values go from 0 to 1.

Problem 6) Improper integrals (10 points)

Which of the following improper integral are convergent? In each case either state that it is not convergent or compute the limit explicitly.

a) (2 points) $\int_1^\infty \sin(x) \, dx$

b) (2 points) $\int_1^\infty \frac{1}{x^3} \, dx$

c) (2 points) $\int_1^\infty \frac{1}{x^{1/3}} \, dx$

d) (2 points) $\int_0^1 \frac{1}{x^3} \, dx$

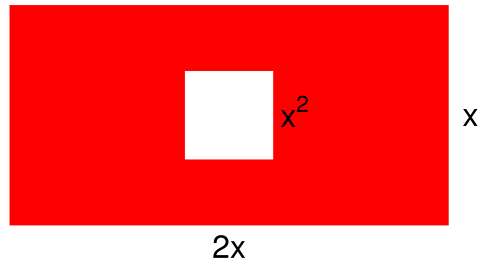
e) (2 points) $\int_0^1 \frac{1}{x^{1/3}}\, dx$

Problem 7) Extrema (10 points)

We want to find the maximal area of a rectangle of length $2x$ and height x in which a square hole of length x^2 has been taken out. The area function is

$$f(x) = 2x^2 - x^4 \ .$$

Use the second derivative test to locate the maximum.



Problem 8) Integration by parts (10 points)

a) (5 points) Compute

$$\int x^5 e^x \ dx \ .$$

b) (5 points) Evaluate the following integral. As always, $\log(x)$ is the natural log.

$$\int \log(x) x \ dx \ .$$

Problem 9) Substitution (10 points)

a) (5 points)

$$\int \frac{\sin(\log(x))}{x} \ dx \ .$$

b) (5 points)

$$\int (1-x^2)^{-5/2} dx \ .$$

Problem 10) Advanced integration (10 points)

a) (5 points) Integrate

$$\int \frac{1}{(x+1)(x+3)(x-2)(x-1)} \ dx \ .$$

b) (5 points) Use the magic **trig substitution box** to find the anti-derivative:

$$\int \frac{4dx}{\sin^3(x)} \ .$$

Here is the magic box:

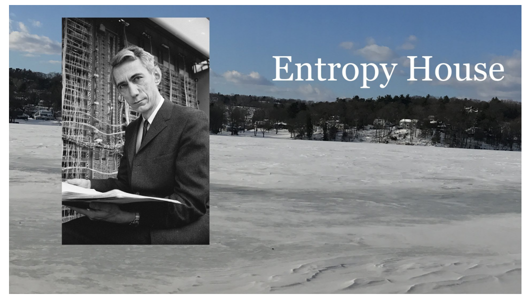
$u = \tan(x/2)$
$dx = \frac{2du}{(1+u^2)}$
$\sin(x) = \frac{2u}{1+u^2}$
$\cos(x) = \frac{1-u^2}{1+u^2}$

Problem 11) PDF's and CDF's. (10 points)

If $f(x)$ is a PDF, then

$$S = -\int_{\infty}^{\infty} f(x)\log(f(x))\,dx$$

is called the **entropy** of f.
What is the entropy of the exponential distribution, given by the function which is 0 for negative x and e^{-x} for $x \geq 0$?



The Entropy house in Winchester, MA on Mystic Lake, where Claude Shannon, the father of information theory lived. Photo: Oliver Knill, 2018.

Problem 12) Which integration method?(10 points)

a) (5 points) Find the anti-derivative of

$$\int e^{e^x} e^x \, dx \; .$$

b) (5 points) And what is the anti-derivative of

$$\int (\log(x))^2 x \, dx \; .$$

Problem 13) Applications (10 points)

a) (2 points) The CDF of the PDF. What is the CDF of the function that is given as $f(x) = 1/x^2$ if $x \geq 1$ and $f(x) = 0$ else? [Give the function for $x \geq 1$]

b) (2 points) If x is the Midi number, then $f(x) = 440e^{(x-69)/12}$ is called the [give the expression, one word.]

c) (2 points) If $f(x)$ is a PDF, then $M_n = \int_{-\infty}^{\infty} x^n f(x)\, dx$ is called a [give the expresion, one word.]

d) (2 points) For the family of functions $f_c(x) = c\cos(x)$, there is a catastrophe at $c =$ [give a number, one number.]

e) (2 points) If $(3, 5)$ and $(2, 7)$ are two data points, the line $y = mx$ which is the best fit minimizes the function [Give a function $f(m) = ...$]

Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

5/7/2021: Final Practice B

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 9 AM on May 8th. Do not communicate with anybody related to the class during the exam period and with nobody at all about the exam.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points). No justifications are needed.

1) T F $\cos(17\pi/4) = \sqrt{2}/2$.

2) T F The tangent function is monotonically increasing on the open interval $(-\pi/2, \pi/2)$.

3) T F The arccot function is monotonically increasing from $\pi/4$ to $3\pi/4$.

4) T F If f is a probability density function, then $\int_{-\infty}^{\infty} f(x)\, dx = 0$

5) T F $\frac{d}{dx} e^{\log(x)} = 1$.

6) T F If $f''(0) = -1$ then f has a local maximum at $x = 0$.

7) T F The improper integral $\int_{-1}^{1} 1/|x|\, dx$ is finite.

8) T F The function $-\cos(x) - x$ has a root in the interval $(-100, 100)$.

9) T F If a function f has a local maximum in $(0, 1)$ then it also has a local minimum in $(0, 1)$.

10) T F The anti derivative of $1/(1 - x^2)$ is equal to $\arctan(x)$.

11) T F The function $f(x) = (e^x - e^{2x})/(x - x^2)$ has the limit 1 as x goes to zero.

12) T F If you listen to the sound $e^{-x} \sin(10000x)$, then it gets louder and louder as time goes on.

13) T F The function $f(x) = e^{x^2}$ has a local minimum at $x = 0$

14) T F The function $f(x) = (x^{55} - 1)/(x - 1)$ has the limit 1 for $x \to 1$.

15) T F If the total cost $F(x)$ of an entity is extremal at x, then we have a break even point $f(x) = g(x)$.

16) T F The value $\int_{-\infty}^{\infty} x f(x)\, dx$ is called the expectation of the PDF f.

17) T F The trapezoid rule is an integration method in which the left and right Riemann sum are averaged.

18) T F $\tan(\pi/3) = \sqrt{3}$.

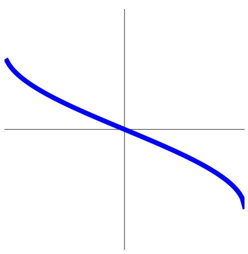
19) T F A Newton step for the function f is $T(x) = x + \frac{f(x)}{f'(x)}$.

20) T F $\sin(\arctan(1)) = \sqrt{3}$.

Problem 2) Matching problem (10 points) No justifications needed

(5 points) Match the functions names with their graphs (1-4) their derivatives (A-D) (middle row) and second derivatives (a-d) (last row).

Function	fill in 1)-4)	fill in A)-D)	fill in a)-d)
$\sin(x)/x$			
$\tan(x)$			
$\arcsin(x)$			
$1/(1+x^2)$			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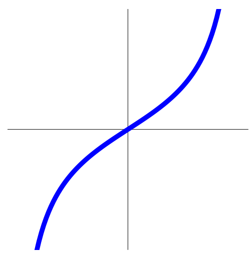
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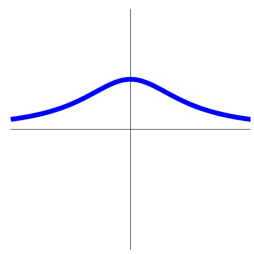
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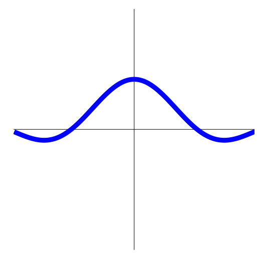
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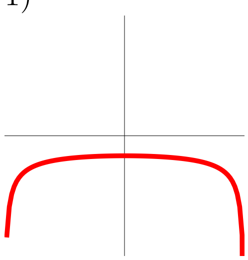
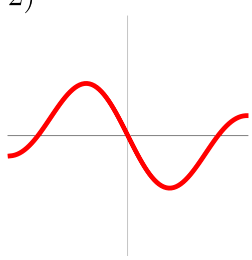

A)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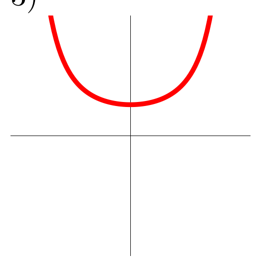
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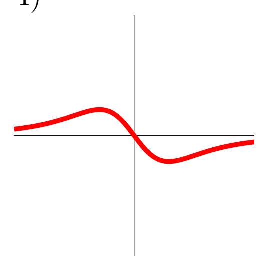
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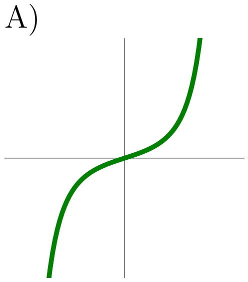
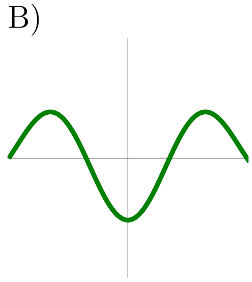

a)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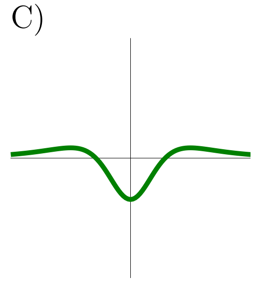
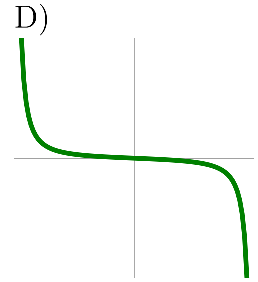

c)

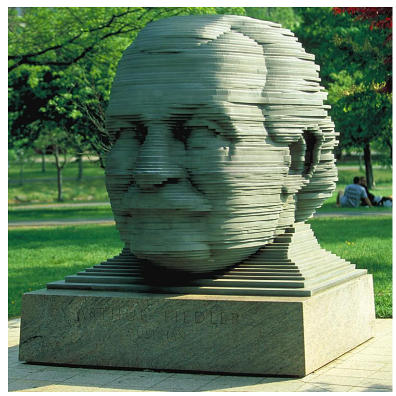

d)

(5 points) Which of the following limits exists in the limit $x \to 0$.

Function	exists	does not exist
$\sin^4(x)/x^4$		
$1/\log\|x\|$		
$\arctan(x)/x$		
$\log\|x\|/(x-1)$		
$\cos(x)/(x-1)$		
$(x^{10}-1)/(x-1)$		

Problem 3) Matching or short answer problem (10 points). No justifications are needed.

a) (4 points) On the Boston Esplanade is a sculpture of **Arthur Fiedler** (1894-1979) a long-time conductor of the Boston Pops Orchestra. His head is sliced into thin slices. Assume that the thickness of each level is $h = 1.5$ inch and the area of each of the 100 slices k is $A(k)$. Which formula gives the volume of the head? (One applies.)



Formula	Check if true
$1.5[A(1) + \cdots + A(100)]$	
$\frac{1}{1.5}[A(1) + \cdots + A(100)]$	

Formula	Check if true
$1.5[\frac{1}{A(1)} + \cdots + \frac{1}{A(100)}]$	
$\frac{1.5}{100}[A(1) + \cdots + A(100)]$	

b) (4 points) The summer has arrived on May 12 2014 for a day before it cooled down again. Harvard students enjoy the **Lampoon pool** that day in front of the **Lampoon castle**. Assume the water volume at height z is $V(z) = 1 + 5z - \cos(z)$. Assume water evaporates at a rate of $V'(z) = -1$ gallon per day. How fast does the water level drop at $z = \pi/2$ meters? Check the right answer: (one applies)

Rate	Check if true
-6	
$-1/6$	

Rate	Check if true
-4	
$-1/4$	

c) (2 points) Speaking of weather: the temperature on April 28, 2021 in Cambridge was 53 degrees Fahrenheit. Four days before, on April 24, the temperature had been 70 degrees and have us dream about **summer time**. Which of the following theorems assures that there was a moment during the night of April 24 to May 28 that the temperature was exactly 61.5 degrees? (One applies.)

Theorem	check if true	Theorem	check if true
Mean value theorem		Intermediate value theorem	
Fermat's theorem		Bolzano theorem	

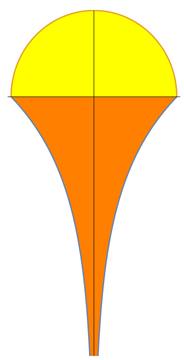
Problem 4) Area computation (10 points)

Find the area enclosed by the graphs of the functions

$$f(x) = \log |x|$$

and

$$g(x) = \sqrt{1 - x^2} \; .$$



Problem 5) Volume computation (10 points)

The lamps near the front entrance of the **Harvard Malkin Athletic Center** (MAC) have octagonal cross sections, where at height z, the area is

$$A(z) = 2(1 + \sqrt{2})(1 + z)^2$$

with $0 \leq z \leq 3$. What is the volume of the lamp?

Problem 6) Improper integrals (10 points)

Which of the following limits $R \to \infty$ exist? If the limit exist, compute it.

a) (2 points) $\int_1^R \sin(2\pi x)\, dx$

b) (2 points) $\int_1^R \frac{1}{x^2}\, dx$

c) (2 points) $\int_1^R \frac{1}{\sqrt{x}}\, dx$

d) (2 points) $\int_1^R \frac{1}{1+x^2}\, dx$

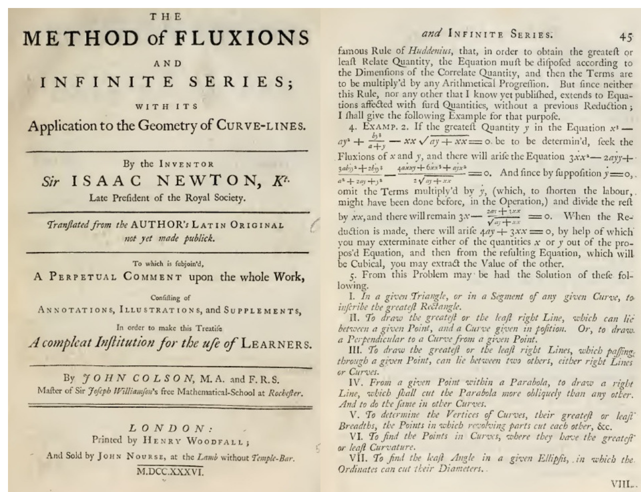
e) (2 points) $\int_1^R x\, dx$

Problem 7) Extrema (10 points)

In Newton's masterpiece **"The Method of Fluxions"** on the bottom of page 45, Newton asks: "In a given triangle or in a segment of any given curve, to inscribe the greatest rectangle." Lets be more specific and find rectangle with largest area

$$A = xy$$

in the triangle given by the x-axes, y-axes and line $y = 2 - 2x$. Use the second derivative test to make sure you have found the maximum.



THE
METHOD of FLUXIONS
AND
INFINITE SERIES;
WITH ITS
Application to the Geometry of CURVE-LINES.

By the INVENTOR
Sir ISAAC NEWTON, Kt.
Late Preſident of the Royal Society.

Tranſlated from the AUTHOR'S LATIN ORIGINAL
not yet made publick.

To which is ſubjoin'd,
A PERPETUAL COMMENT upon the whole Work,
Conſiſting of
ANNOTATIONS, ILLUSTRATIONS, and SUPPLEMENTS,
In order to make this Treatiſe
A compleat Inſtitution for the uſe of LEARNERS.

By *JOHN COLSON*, M.A. and F.R.S.
Maſter of Sir *Joſeph Williamſon*'s free Mathematical-School at *Rocheſter*.

LONDON:
Printed by HENRY WOODFALL;
And Sold by JOHN NOURSE, at the *Lamb* without *Temple-Bar*.
M.DCC.XXXVI.

and INFINITE SERIES. 45

famous Rule of *Huddenius*, that, in order to obtain the greateſt or leaſt Relate Quantity, the Equation muſt be diſpoſed according to the Dimenſions of the Correlate Quantity, and then the Terms are to be multiply'd by any Arithmetical Progreſſion. But ſince neither this Rule, nor any other that I know yet publiſhed, extends to Equations affected with ſurd Quantities, without a previous Reduction; I ſhall give the following Example for that purpoſe.

4. EXAMP. 2. If the greateſt Quantity y in the Equation $x^3 - ay^2 + \frac{by^3}{a+y} - xx\sqrt{ay+xx} = 0$ be to be determin'd, ſeek the Fluxions of x and y, and there will ariſe the Equation $3\dot{x}x^2 - 2a\dot{y}y + \frac{3ab\dot{y}y^2 + 2b\dot{y}y^3}{a^2+2ay+y^2} - \frac{4a\dot{x}xy + 6\dot{x}x^3 + a\dot{y}x^2}{2\sqrt{ay+xx}} = 0$. And ſince by ſuppoſition $\dot{y} = 0$, omit the Terms multiply'd by $\dot{y}$, (which, to ſhorten the labour, might have been done before, in the Operation,) and divide the reſt by $\dot{x}x$, and there will remain $3x - \frac{2ay + 3xx}{\sqrt{ay+xx}} = 0$. When the Reduction is made, there will ariſe $4ay + 3xx = 0$, by help of which you may exterminate either of the quantities x or y out of the propos'd Equation, and then from the reſulting Equation, which will be Cubical, you may extract the Value of the other.

5. From this Problem may be had the Solution of theſe following.

I. *In a given Triangle, or in a Segment of any given Curve, to inſcribe the greateſt Rectangle.*

II. *To draw the greateſt or the leaſt right Line, which can lie between a given Point, and a Curve given in poſition. Or, to draw a Perpendicular to a Curve from a given Point.*

III. *To draw the greateſt or the leaſt right Lines, which paſſing through a given Point, can lie between two others, either right Lines or Curves.*

IV. *From a given Point within a Parabola, to draw a right Line, which ſhall cut the Parabola more obliquely than any other. And to do the ſame in other Curves.*

V. *To determine the Vertices of Curves, their greateſt or leaſt Breadths, the Points in which revolving parts cut each other,* &c.

VI. *To find the Points in Curves, where they have the greateſt or leaſt Curvature.*

VII. *To find the leaſt Angle in a given Ellipſis, in which the Ordinates can cut their Diameters.*

VIII.

Problem 8) Integration by parts (10 points)

a) (5 points) Find

$$\int (1 + x + x^2 + x^3 + x^4)(\sin(x) + e^x)\, dx\ .$$

b) (5 points) Find

$$\int \log(x) \frac{1}{x^2}\, dx\ .$$

Problem 9) Substitution (10 points)

a) (5 points) **"One,Two,Three,Four Five, once I caught a fish alive!"**

$$\int \frac{(1+2x+3x^2+4x^3+5x^4)}{(1+x+x^2+x^3+x^4+x^5)} \, dx \, .$$

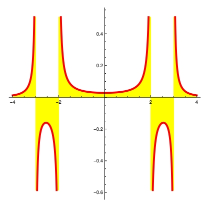
b) (5 points) A **"Trig Trick-or-Treat"** problem:

$$\int (1-x^2)^{-3/2} + (1-x^2)^{-1/2} + (1-x^2)^{1/2} \, dx \, .$$

Problem 10) Partial fractions (10 points)

Integrate

$$\int_{-1}^{1} \frac{1}{(x+3)(x+2)(x-2)(x-3)} \, dx \, .$$



The graph of the function is shown to the right.

Lets call it the **friendship graph**.

Problem 11) Chain rule. (10 points)

a) Find the derivative of

$$f(x) = (\sin(7x + x\cos(x)) - 3x) \, .$$

in general.
b) Now avaluate at $x = 0$.

Problem 12) Various integration problems (10 points)

a) (2 points) $\int_0^{2\pi} 2\cos^2(x) - \sin(x) \, dx$

b) (2 points) $\int x^2 e^{3x} \, dx$

c) (2 points) $\int_1^{\infty} \frac{1}{(x+2)^2} \, dx$

d) (2 points) $\int \sqrt{x} \log(x) \, dx$

e) (2 points) $\int_1^{e} \log(x)^2 \, dx$

Problem 13) Applications (10 points)

a) (2 points) [**Agnesi density**]

The CDF of the PDF $f(x) = \pi^{-1}/(1+x^2)$ is

b) (2 points) [**Piano man**]

The upper hull of $f(x) = x^2 \sin(1000x)$ is the function

c) (2 points) [**Rower's wisdom**]

If f is power, F is work and $g = F/x$ then $f = g$ if and only if $g'(x) =$

d) (2 points) [**Catastrophes**]

For $f(x) = c(x-1)^2$ there is a catastrophe at $c =$

e) (2 points) [**Randomness**]

We can use chance to compute integrals. It is called the method.

5/7/2021: Final Practice C

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 9 AM on May 8th. Do not communicate with anybody related to the class during the exam period and with nobody at all about the exam.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points). No justifications are needed.

1) T F $\frac{d}{dx}e^{e^x} = e^x$.

2) T F A function f which is concave down at 0 satisfies $f''(0) \leq 0$.

3) T F The integral $\int_{1/2}^{1} \log(x)\ dx$ is positive. Here $\log(x) = \ln(x)$ is the natural log.

4) T F The function $x + \sin(\cos(\sin(x)))$ has a root in the interval $(-10, 10)$.

5) T F The function $x(1-x) + \sin(\sin(x(1-x)))$ has a maximum or minimum inside the interval $(0, 1)$.

6) T F The derivative of $1/(1+x^2)$ is equal to $\arctan(x)$.

7) T F The limit of $\sin^{100}(x)/x^{100}$ for $x \to 0$ exists and is equal to 100.

8) T F The function $f(x) = (1-e^x)/\sin(x)$ has the limit 1 as x goes to zero.

9) T F The frequency of the sound $\sin(10000x)$ is higher than the frequency of $\sin(\sin(3000x))$.

10) T F The function $f(x) = \sin(x^2)$ has a local minimum at $x = 0$

11) T F The function $f(x) = (x^5 - 1)/(x-1)$ has a limit for $x \to 5$.

12) T F The average cost $g(x) = F(x)/x$ of an entity is extremal at x for which $f(x) = g(x)$. Here, $f(x)$ denotes the marginal cost and $F(x)$ the total cost.

13) T F The mean of a probability density function is defined as $\int f(x)\ dx$.

14) T F The differentiation rule $(f(x)^{g(x)})' = (f'(x))^{g(x)} g'(x)$ holds for all differentiable functions f, g.

15) T F $\sin(5\pi/6) = 1/2$.

16) T F Hôpital's rule assures that $\sin(10x)/\tan(10x)$ has a limit as $x \to 0$.

17) T F A Newton step for the function f is $T(x) = x - \frac{f'(x)}{f(x)}$.

18) T F A minimum x of a function f is called a catastrophe if $f'''(x) < 0$.

19) T F The fundamental theorem of calculus implies $\int_{-1}^{1} g'(x)\ dx = g(1) - g(-1)$ for all differentiable functions g.

20) T F If f is a differentiable function for which $f'(x) = 0$ everywhere, then f is constant.

Problem 2) Matching problem (10 points) No justifications needed

a) (2 points) One of three statements A)-C) is not the part of the fundamental theorem of calculus. Which one?

A)	$\int_0^x f'(t)\, dt = f(x) - f(0)$
B)	$\frac{d}{dx} \int_0^x f(t)\, dt = f(x)$
C)	$\int_a^b f(x)\, dx = f(b) - f(a)$

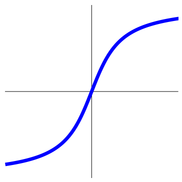
b) (3 points) Biorythms can be fascinating for small kids, giving them a first exposure to trig functions and basic arithmetic. The "theory" tells that there are three functions $p(x) = \sin(2\pi x/23)$ (Physical) $e(x) = \sin(2\pi x/28)$ (Emotional) and $i(x) = \sin(2\pi x/33)$ (Intellectual), where x is the number of days since your birth. Assume **Tuck**, the pig you know from the practice exams, is born on October 10, 2005. Today, on May 11, 2014, it is 2670 days old. Its biorythm is $E = 0.7818, P = -0.299, I = -0.5406$. It is a happy fellow, tired, but feeling a bit out of spirit, like the proctor of this exam feels right now. Which of the following statements are true?

Check if true	
	i) One day old Tuck had positive emotion, intellect and physical strength.
	ii) Among all cycles, the physical cycle takes the longest to repeat.
	iii) Comparing with all cycles, the physical increases fastest at birth.

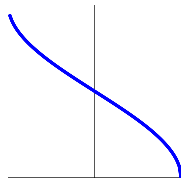
c) (4 points) Name the statements:

$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ is called the	.
Rule $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$ is called	.
$\int_0^x f'(t)\, dt = f(x) - f(0)$ is called	.
The PDF $f(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ is called the	

d) (1 point) Which of the following graphs belongs to the function $f(x) = \arctan(x)$?



1)



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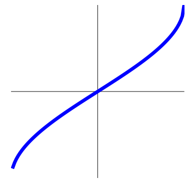
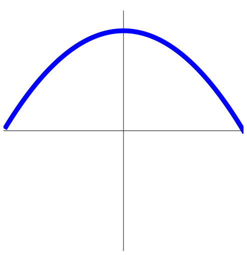


3)

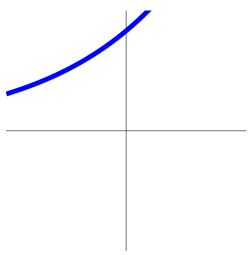
Problem 3) Matching or short answer problem (10 points). No justifications are needed.

a) (4 points) Match the functions (a-d) (top row) with their derivatives (1-4) (middle row) and second derivatives (A-D) (last row).

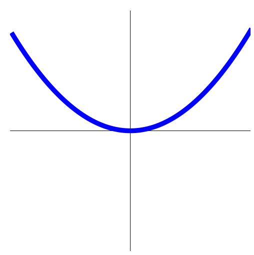
Function a)-d)	Fill in 1)-4)	Fill in A)-D)
graph a)		
graph b)		
graph c)		
graph d)		



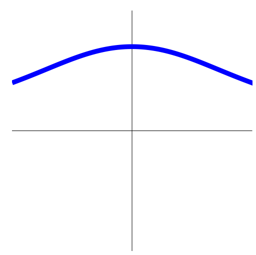
a)



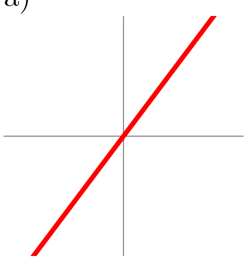
b)



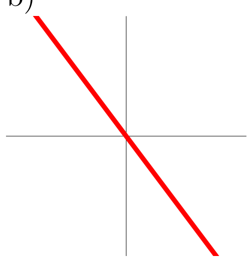
c)



d)



1)



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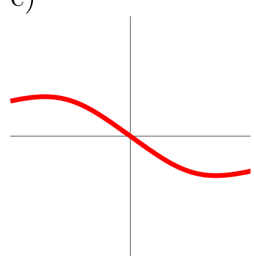

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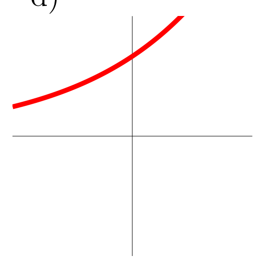

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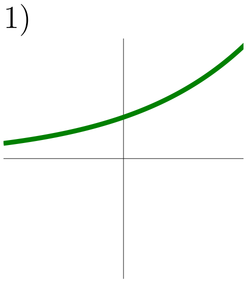
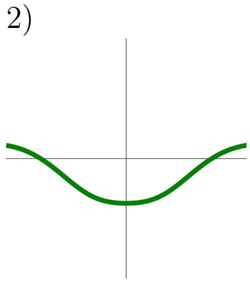
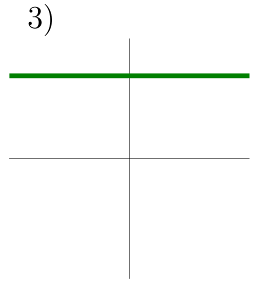


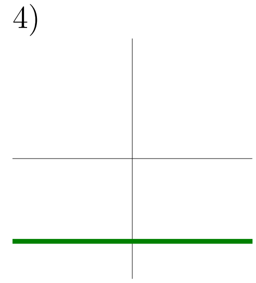
A)



B)



C)

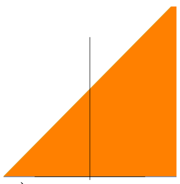


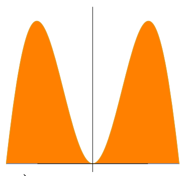
D)

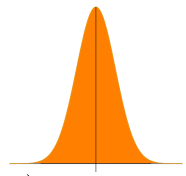
b) (4 points) Match the following integrals with the areas in the figures:

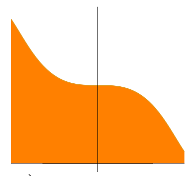
Integral	Enter 1-4
$\int_{-\pi}^{\pi} x\sin(x)\,dx.$	
$\int_{-\pi}^{\pi} \exp(-x^2)\,dx.$	

Integral	Enter 1-4
$\int_{-\pi}^{\pi} \pi + x\,dx.$	
$\int_{-\pi}^{\pi} 1-\sin(x^3/\pi^3)\,dx.$	


1)


2)


3)

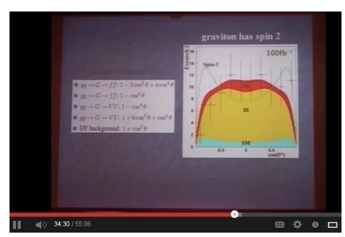

4)

c) (2 points) Name two different numerical integration methods. We have seen at least four.

Your first method	
Your second method	

Problem 4) Area computation (10 points)

A slide in a lecture of Harvard physicist **Lisa Randall** shows the area between two functions. Lisa is known for her theory of "branes" which can explain why gravity is so much weaker than electromagnetism. Assist Lisa and write down the formula for the area between the graphs of $1-\cos^2(x)$ and $1-\cos^4(x)$, where $0 \leq x \leq \pi$. Find the area.



Lisa Randall - Extra Dimensional Particle Resonances at the LHC (1/3)

Hint. Lisa already knows the identity

$$\cos^2(x) - \cos^4(x) = \cos^2(x)(1-\cos^2(x)) = \cos^2(x)\sin^2(x)\ .$$

Problem 5) Volume computation (10 points)

Find the volume of the solid of revolution for which the radius at height z is

$$r(z) = \sqrt{z\log(z)}$$

and for which z is between 1 and 2. Here, log is the natural log. Naturalmente!

Problem 6) Improper integrals (10 points)

a) (5 points) Find the integral or state that it does not exist

$$\int_1^\infty \frac{7}{x^{3/4}}\ dx\ .$$

b) (5 points) Find the integral or state that it does not exist

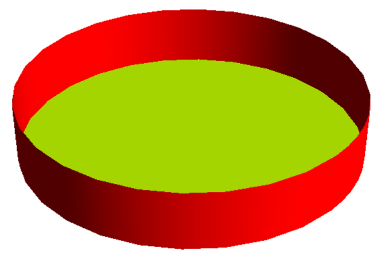
$$\int_1^\infty \frac{13}{x^{5/4}}\ dx\ .$$

Problem 7) Extrema (10 points)

A **candle holder** of height y and radius x is made of aluminum. Its total surface area is $2\pi xy + \pi x^2 = \pi$ (implying $y = 1/(2x) - x/2$). Find x for which the volume

$$f(x) = x^2 y(x)$$

is maximal.



Problem 8) Integration by parts (10 points)

a) (5 points) Find

$$\int (x+5)^3 \sin(x-4)\ dx\ .$$

b) (5 point) Find the indefinite integral

$$\int e^x \cos(2x)\ dx\ .$$

Don't get dizzy when riding this one.

Problem 9) Substitution (10 points)

a) (3 points) Solve the integral $\int \log(x^3) x^2\ dx$.

b) (4 points) Solve the integral $\int x\cos(x^2)\exp(\sin(x^2))\,dx$.

c) (3 points) Find the integral $\int \sin(\exp(x))\exp(x)\,dx$.

Problem 10) Partial fractions (10 points)

a) (5 points) Find the definite integral

$$\int_1^5 \frac{1}{(x-2)(x-3)(x-4)}\,dx\ .$$

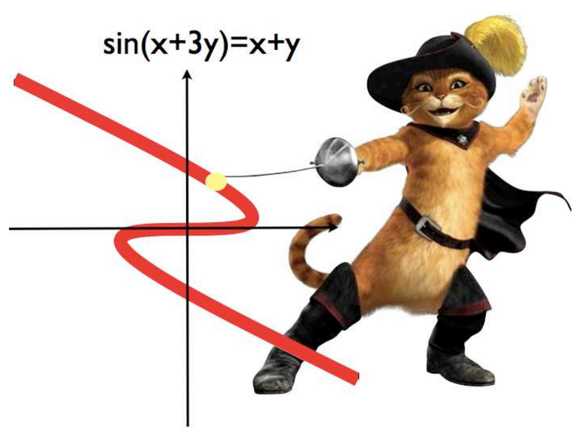
(Evaluate the absolute values $\log|\cdot|$ in your answer. The improper integrals exist as a Cauchy principal value).

b) (5 points) Find the indefinite integral

$$\int \frac{1}{x(x-1)(x+1)(x-2)}\,dx\ .$$

Problem 11) Chain rule. (10 points)

a) Find the derivative of $f(g(x))$, where $f(x) = \sin(\pi x)$ and $g(x) = x^4 + 3x$. b) Let us look at a specific point x. While x is unknown, you know $g(x) = 4$ and $g'(x) = 7$. What is $\frac{d}{dx}f(g(x))$ at this point?



Problem 12) Various integration problems (10 points)

Find the anti-derivatives of the following functions:

a) (2 points) $f(x) = x\log(x) + \frac{1}{1+x^2}$.

b) (3 points) $f(x) = \frac{2x}{x^2+1} + \frac{1}{x^2-4}$.

c) (2 points) $f(x) = \sqrt{16 - x^2} + \frac{1}{\sqrt{1-x^2}}$.

d) (3 points) $f(x) = \log(x) + \frac{1}{x\log(x)}$.

Problem 13) Applications (10 points)

a) (2 points) Find the CDF $\int_0^x f(t)\, dt$ for the PDF which is $f(x) = \exp(-x/3)/3$ for $x \geq 0$ and 0 for $x < 0$.

b) (2 points) Perform a single Newton step for the function $f(x) = \sin(x)$ starting at $x = \pi/3$.

c) (2 points) Check whether the function $f(x) = 1/(2x^2)$ on the real line $(-\infty, \infty)$ is a probability density function.

d) (2 points) A rower produces the power $P(t)$ is $\sin^2(10t)$. Find the energy $\int_0^{2\pi} P(t)\, dt$ when rowing starting at time $t = 0$ and ending at $t = 2\pi$.

e) (2 points) What is the frequency of the Midi number 10?

Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

5/7/2021: Final Practice D

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 9 AM on May 8th. Do not communicate with anybody related to the class during the exam period and with nobody at all about the exam.

1		20
2		10
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5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points). No justifications are needed.

1) T F The definite integral $\int_0^{2\pi} \sin^2(5x)\, dx$ is zero.

2) T F The intermediate value theorem assures that the function $\exp(\sin(x))$ has a root in the interval $(0, 2\pi)$.

3) T F $\frac{d}{dx} \cos(4x) = -4\sin(4x)$.

4) T F If $f''(1) < 0$ then 1 is a local maximum of f.

5) T F The derivative of $1/x$ is $\log(x)$ for all $x > 0$.

6) T F The limit of $\sin(3x)/(5x)$ for $x \to 0$ exists and is equal to $3/5$.

7) T F The function $(e^t - 1)/t$ has the limit 1 as t goes to zero.

8) T F The derivative of $f(f(x))$ is $f'(f'(x))$ for any differentiable function f.

9) T F A monotonically increasing function f has no point x, where $f'(x) < 0$.

10) T F The function $f(x) = \exp(-x^2)$ has an inflection point x somewhere on the real line.

11) T F The function $f(x) = (1 - x^3)/(1 + x)$ has a limit for $x \to -1$.

12) T F If we know the marginal cost for all quantities x as well as the total cost for $x = 1$ we know the total cost for all x.

13) T F The function f which satisfies $f(x) = 0$ for $x < 0$ and $f(x) = e^{-x}$ for $x \geq 0$ is a probability density function.

14) T F The differentiation rule $(f \cdot g)' = f'(g(x)) \cdot g'(x)$ holds for all differentiable functions f, g.

15) T F Hôpital's rule assures that $\cos(x)/\sin(x)$ has a limit as $x \to 0$.

16) T F A Newton step for the function f is $T(x) = x - \frac{f(x)}{f'(x)}$.

17) T F The family of functions $f_c(x) = cx^2$ where c is a parameter has a catastrophe at $x = 0$.

18) T F The fundamental theorem of calculus implies $\int_{-x}^{x} f'(t)\, dt = f(x) - f(-x)$ for all differentiable functions f.

19) T F If f is a smooth function for which $f''(x) = 0$ everywhere, then f is constant.

20) T F The function $f(x) = \sin(x)/(1 - \cos(x))$ can be assigned a value $f(0)$ such that $f(x)$ is continuous at 0.

Problem 2) Matching problem (10 points) Only short answers are needed.

We name some important concepts in this course. To do so, please complete the sentences with one or two words. Each question is one point.

$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ is called the		of f.
$f'(x) = 0, f''(x) > 0$ implies that x is a		of f.
The sum $\frac{1}{n}[f(0) + f(1/n) + f(2/n) + \cdots + f((n-1)/n) + f(1)]$ is called a		sum.
If $f(0) = -3$ and $f(4) = 8$, then f has a root on the interval $(0, 4)$ by the		theorem.
There is a point $x \in (0, 1)$ where $f'(x) = f(1) - f(0)$ by the		theorem.
The expansion rate $r'(t)$ can be obtained from $d/dtV(r(t)) = -5$ by the differentiatl rule called		rates.
The anti derivative $\int_{-\infty}^{x} f(t)\ dt$ of a probability density function f is called the		function.
A point x for which $f(x) = 0$ is called a		of f.
A point x for which $f''(x) = 0$ is called an		of f.
At a point x for which $f''(x) > 0$, the function is called		up.

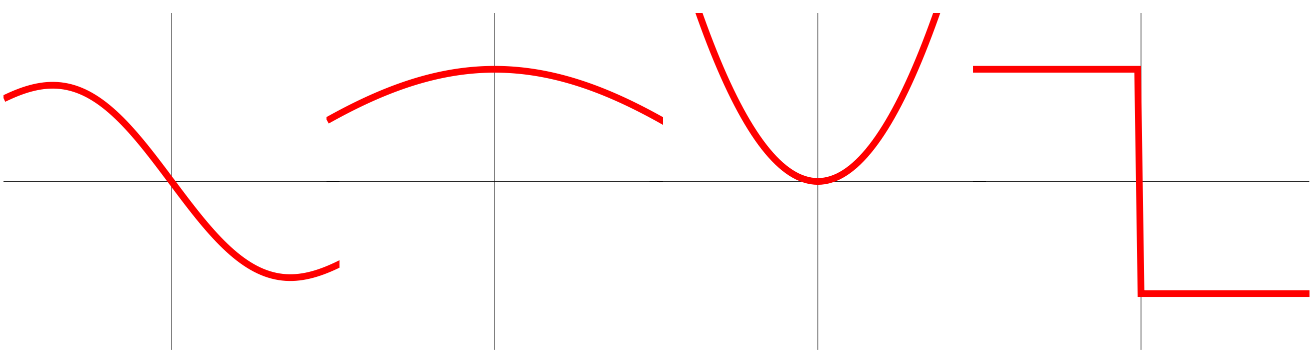
Problem 3) Matching or short answer problem (10 points). No justifications are needed.

a) (4 points) Find the relation between the following functions:

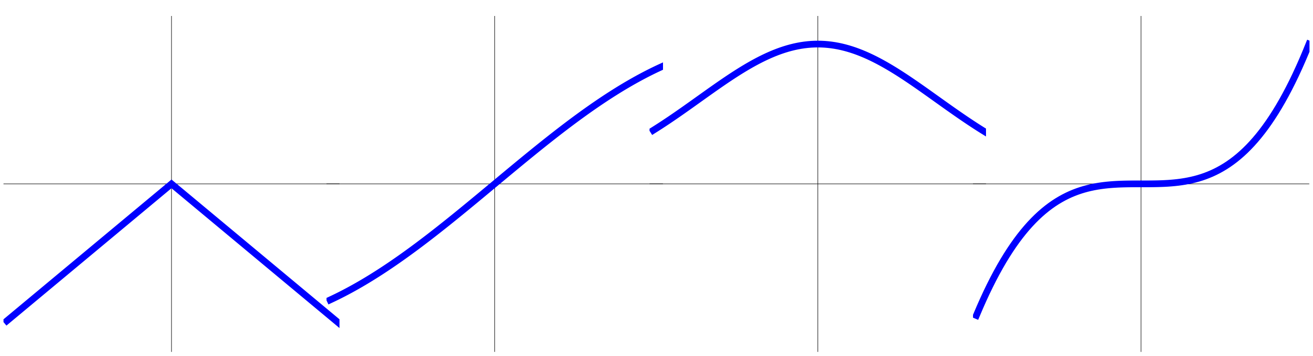
function f	function g	$f = g'$	$g = f'$	none
$\log\|\sin(x)\|$	$\cot(x)$			
$1/\cos^2(x)$	$\tan(x))$			
x^5	$5x^4$			
$1/x^2$	$-1/x$			
$\sin(\log(x))$	$\cos(\log(x))/x$			

b) (3 points) Match the following functions (a-d) with a choice of **anti-derivatives** (1-4).

Function a)-d)	Fill in 1)-4)
graph a)	
graph b)	
graph c)	
graph d)	



a) b) c) d)



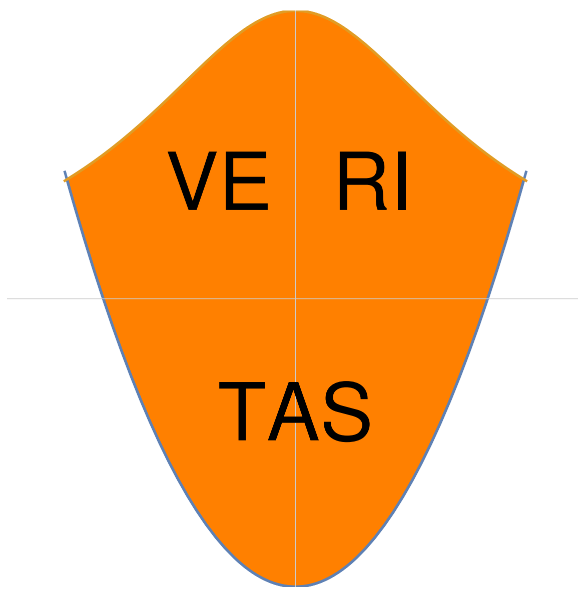
1) 2) 3) 4)

c) (3 points) Find the limits for $x \to 0$

Function f	$\lim_{x \to 0} f(x)$
$x/(e^{2x}-1)$	
$(e^{2x}-1)/(e^{3x}-1)$	
$\sin(3x)/\sin(5x)$	

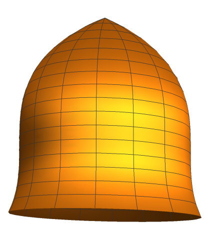
Problem 4) Area computation (10 points)

Find the area of the shield shaped region bound by the two curves $1/(1+x^2)$ and x^2-1.



Problem 5) Volume computation (10 points)

Did you know that there is a scaled copy of the **liberty bell** on the campus of the Harvard business school? Here we compute its volume. Find the volume of the rotationally symmetric solid if the radius $r(z)$ at height z is $r(z) = 8-(z-1)^3$ and the height z of the bell is between 0 and 3.



Problem 6) Improper integrals (10 points)

a) (5 points) Find the integral or state that it does not exist

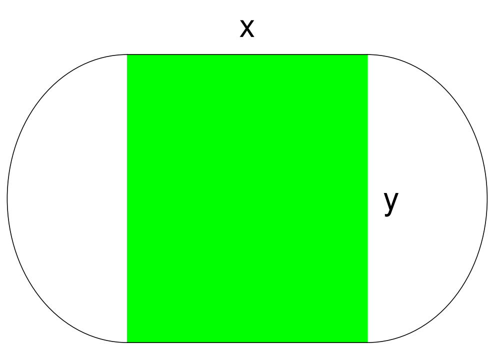
$$\int_1^\infty \frac{1}{x^4} \, dx \; .$$

b) (5 points) Find the integral or state that it does not exist

$$\int_1^\infty \frac{1}{x^{3/2}} \, dx \; .$$

Problem 7) Extrema (10 points)

The Harvard stadium has a track which encloses a rectangular field of dimensions x, y. The circumference of the track is $400 = 2\pi y + 2x$ and is fixed. We want to maximize the area xy for a play field. Which x achieves this?



Problem 8) Integration by parts (10 points)

Find the antiderivative:

$$\int (x-1)^4 \exp(x+1) \, dx \; .$$

Problem 9) Substitution (10 points)

a) (3 points) Solve the integral $\int e^{x^2} 2x \, dx$.

b) (3 points) Solve the integral $\int 2x \log(x^2)\, dx$.

c) (4 points) Find the integral $\int e^{-2e^x} e^x\, dx$.

Problem 10) Partial fractions (10 points)

a) (5 points) Find the definite integral

$$\int_1^5 \frac{1}{(x-4)(x-2)}\, dx .$$

b) (5 points) Find the indefinite integral

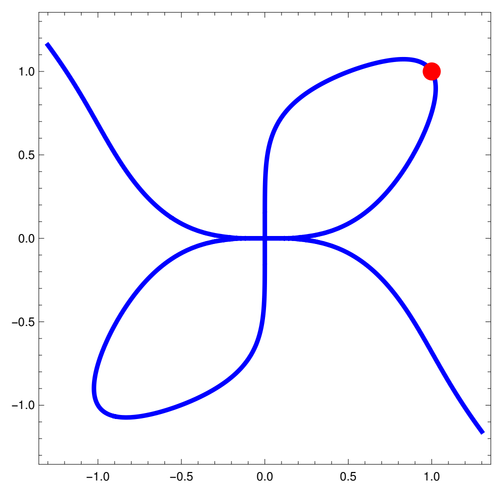
$$\int \frac{1}{(x-1)(x-3)(x-5)}\, dx .$$

Problem 11) Chain rule (10 points)

The coordinates of a car on a freeway intersection are $x(t)$ and $y(t)$. Use the chain rule to differentiate

$$f(t) = x(t)^7 + y(t)^7 - 2x(t)y(t)^2 .$$

Assume we know $x'(0) = 3$ and $x(0) = 1, y(0) = 1$. Find the derivative $y'(t)$.



Problem 12) Various integration problems (10 points)

Find the anti-derivatives of the following functions:

a) (2 points) $f(x) = \sin^5(x) \cos(x)$.

b) (3 points) $f(x) = \frac{1}{x^2+1} + \frac{1}{x^2-1}$.

c) (2 points) $f(x) = \sqrt{1-x^2} + \frac{1}{\sqrt{1-x^2}}$.

d) (3 points) $f(x) = \log(x) + \frac{1}{\log(x)}$.

Problem 13) Applications (10 points)

a) (5 points) We know the total cost $F(x) = -x^3 + 2x^2 + 4x + 1$ for the quantity x. In order to find the positive **break-even point** x satisfying $f(x) = g(x)$, where $g(x) = F(x)/x$ is the total cost and $f(x) = F'(x)$ is the marginal cost, we do - how sweet it is - find the maximum of the average cost $g(x) = F(x)/x$. Find the maximum!

b) (5 points) We know the **”velocity”**, **”acceleration”** and **”jerk”** as the first second and third derivative of position. The fourth, fifth and sixth derivatives of position as a function of time are called **”snap”**, **”crackle”** and **”pop”** according to characters used in a cereal add. Assume we know the snap $x''''(t) = t$. Find $x(t)$ satisfying $x(0) = x'(0) = x''(0) = 0, x'''(0) = 0$.

Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

5/7/2021: Final Practice E

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 9 AM on May 8th. Do not communicate with anybody related to the class during the exam period and with nobody at all about the exam.

1		20
2		10
3		10
4		10
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6		10
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8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points) No justifications are needed.

1) T F The quantum exponential function $\exp_h(x) = (1+h)^{x/h}$ satisfies $D\exp_h(x) = \exp_h(x)$ for $h > 0$.

2) T F The function $\text{sinc}(x) = \sin(x)/x$ has a critical point at $x = 0$.

3) T F The limit of $1/\log(1/|x|)$ for $x \to 0$ exists.

4) T F The strawberry theorem tells that for any $f(x)$, its anti-derivative $F(x)$ and $g(x) = F(x)/x$ the points $f = g$ are the points where $g' = 0$.

5) T F The function $f(x) = \tan(x)$ has a vertical asymptote at $x = \pi/2$.

6) T F The function $x/(1+x)$ converges to 1 for $x \to \infty$ and has therefore a horizontal asymptote.

7) T F The function $f(x) = \tan(x)$ is odd: it satisfies $f(x) = -f(-x)$.

8) T F The function $\sin^3(x)/x^2$ is continuous on the real line.

9) T F With $Df(x) = f(x+1) - f(x)$ we have $D(fg)(x) = Dfg(x+1) + f(x)Dg(x)$.

10) T F If f has a critical point 0 then f has a minimum or maximum at 0.

11) T F The limit of $[\frac{1}{3+h} - \frac{1}{3}]/h$ for $h \to 0$ is $-1/9$.

12) T F The function $(\cos(x) + \sin(3x))/(\sin(4x) + \cos(3x))$ can be integrated using trig substitution.

13) T F The marginal cost is the anti-derivative of the total cost.

14) T F The cumulative distribution function is the anti-derivative of the probability density function.

15) T F The function $\sqrt{1-x^2}$ can be integrated by a trig substitution $x = \cos(u)$.

16) T F The integral $\int_0^1 1/x^2 \, dx$ is finite.

17) T F The chain rule tells that $d/dx f(g(x)) = f'(x)g'(x)$.

18) T F For the function $f(x) = \sin(100x)$ the hull function is constant.

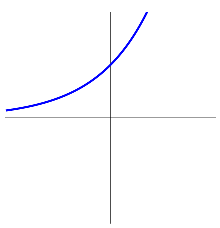
19) T F The trapezoid method is also called Simpson rule.

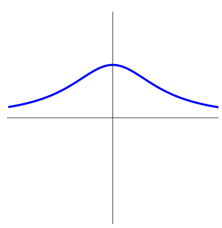
20) T F If $f''(x) > 0$, then the curvature of f is positive.

Problem 2) Matching problem (10 points) No justifications are needed.

Match the following functions with their derivatives.

Function	Fill in the numbers 1-8
graph a)	
graph b)	
graph c)	
graph d)	
graph e)	
graph f)	
graph g)	
graph h)	


a)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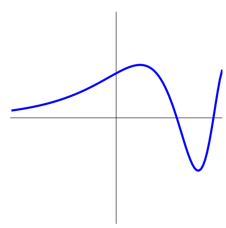
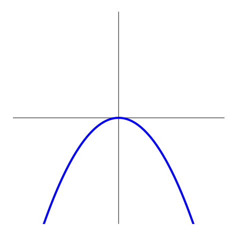
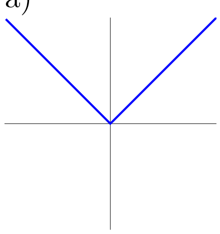

c)


d)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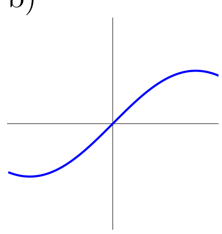
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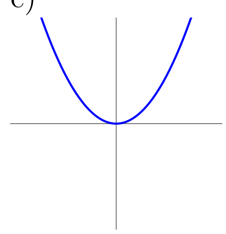
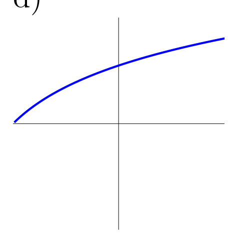

g)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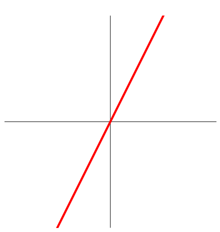
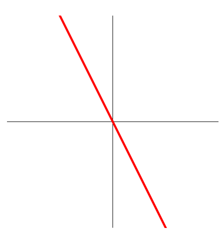
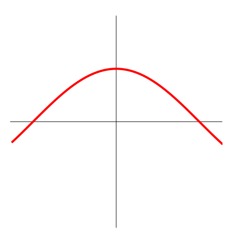

1)


2)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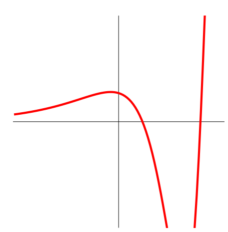
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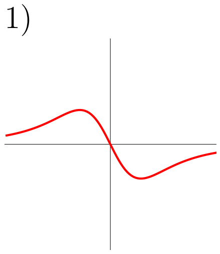
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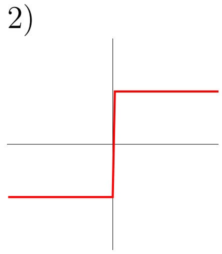
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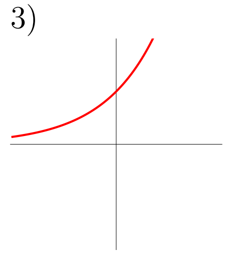
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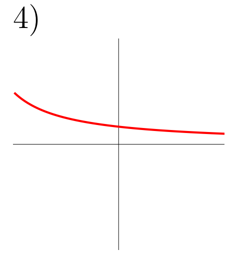
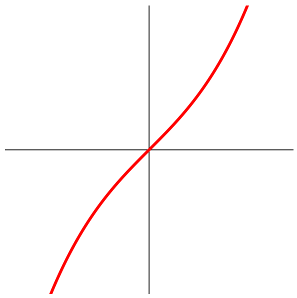

8)

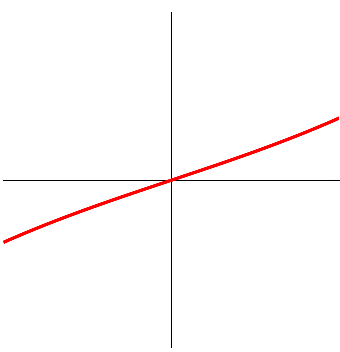
Problem 3) Matching problem (10 points) No justifications are needed.

Here is the graph of a function $f(x)$. Match the following modifications

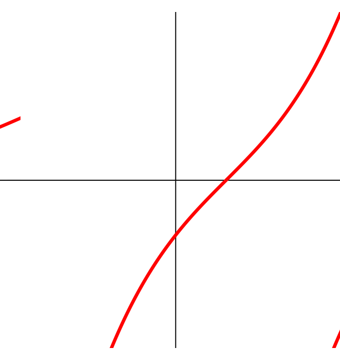


Match the following functions with their graphs.

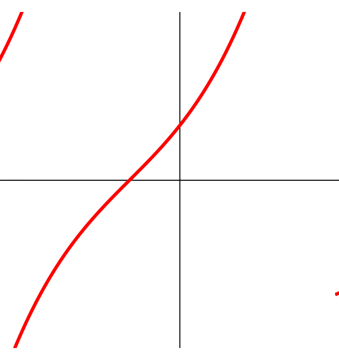
Function	Fill in 1)-8)
$f(x-1)$	
$f'(x)$	
$f(x+1)$	
$f^{-1}(x)$	
$f(x/2)$	
$f(3x)$	
$1/f(x)$	
$f(x)+1$	



1)



2)



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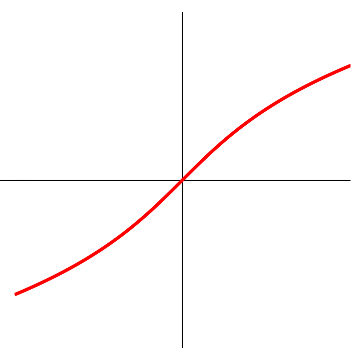

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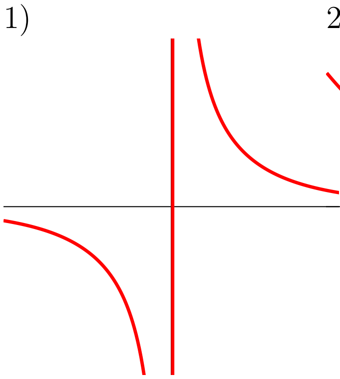

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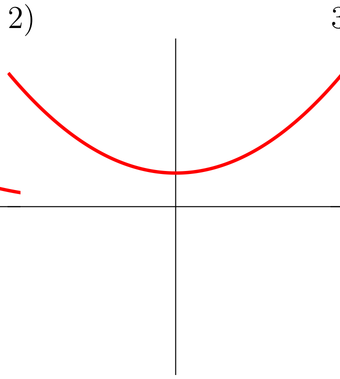

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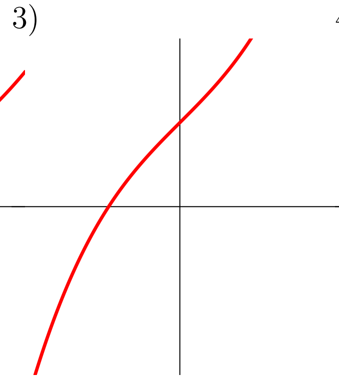

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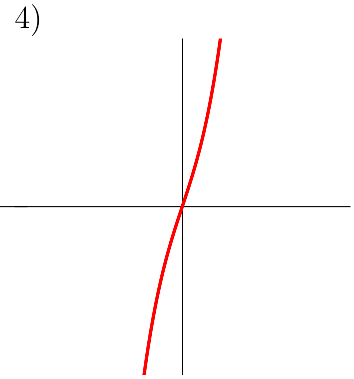
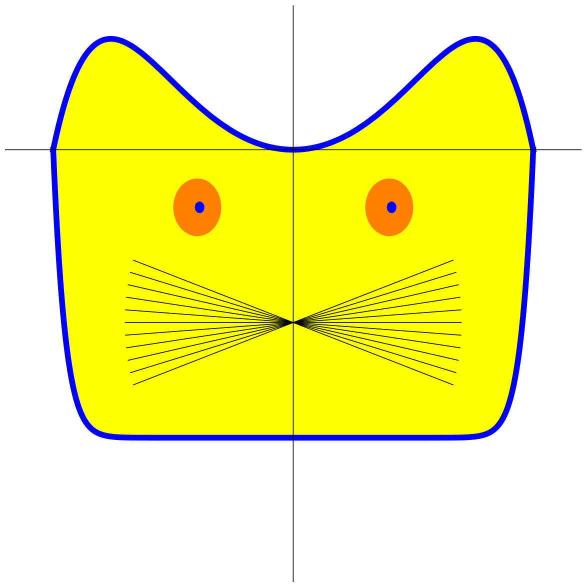


8)

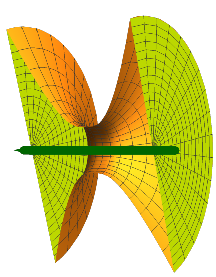
Problem 4) Area computation (10 points)

Find the area of the **cat region** which is the region enclosed by the functions $f(x) = x^{20} - 1$ and $g(x) = x^2 - x^6$. No need to count in the whiskers.



Problem 5) Volume computation (10 points)

We spin the graph of the function $f(x) = \sqrt{1 + |x|^3}$ around the x axes and get a solid of revolution. What is the volume of this solid enclosed between $x = -3$ and $x = 3$? The picture shows half of this sold.



Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals

a) $\int_{-1}^{1} \frac{1}{1+x^2} \, dx$

b) $\int_1^2 x^2 + \sqrt{x}\, dx$

c) $\int_0^{\sqrt{\pi}} \sin(x^2) 2x\, dx$.

d) $\int_0^1 \log(4+x)\, dx$.

Problem 7) Extrema (10 points)

a) (7 points) Analyse the local extrema of the function

$$f(x) = \frac{x}{1+x^2}$$

on the real axes using the second derivative test.

b) (3 points) Are there any global extrema?

Problem 8) Integration by parts (10 points)

a) (5 points) Find the anti-derivative of

$$f(x) = \sin(4x)\cos(3x) .$$

b) (5 points) Find the anti-derivative of

$$f(x) = (x-1)^2 \sin(1+x) .$$

Problem 9) Substitution (10 points)

a) (3 points) Find the integral $\int 3x\sqrt{5x^2-5}\, dx$.

b) (3 points) What is the anti-derivative of $\int \exp(x^2-x)(4x-2)$?

c) (4 points) Evaluate the definite integral

$$\int_0^{\pi/2} \sqrt{1-\cos(x)}\sin(x)\, dx .$$

Problem 10) Partial fractions, Trig substitution (10 points)

a) Solve the integral

$$\int \frac{2-x+x^2}{(1-x)(1+x^2)}$$

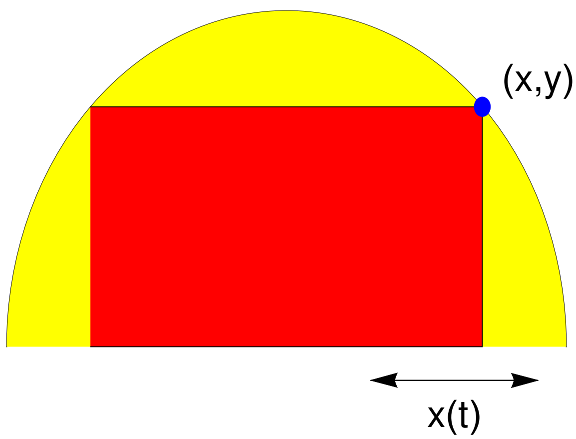
by writing

$$\frac{2-x+x^2}{(1-x)(1+x^2)} = \frac{A}{1+x^2} + \frac{B}{1-x} \ .$$

b) Evaluate the integral $\int \sqrt{1-x^2} x \, dx$.

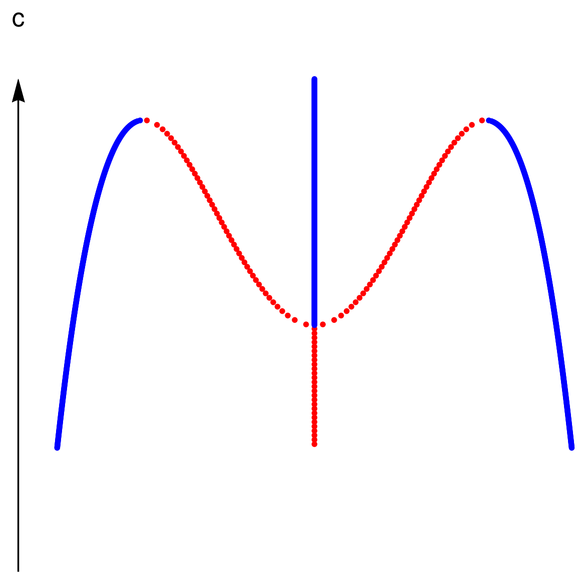
Problem 11) Differentiation rules (10 points)

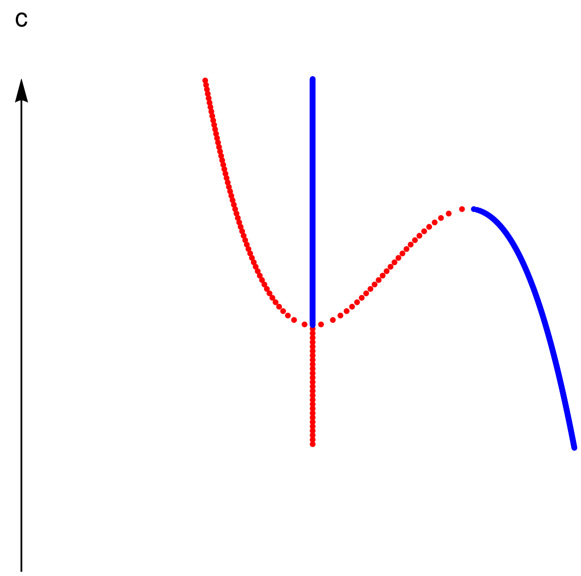
A rectangle with corners at $(-x, 0), (x, 0), (x, y), (-x, y)$ is inscribed in a half circle $x^2 + y^2 = 1$ where $y \geq 0$ is in the upper half plane. We know $x(t) = \cos(t)$ and $y(t) = \sin(t)$. Find the rate of change of the area $2x(t)y(t)$ of the rectangle.



Problem 12) Catastrophes (10 points)

The following two pictures show bifurcation diagrams. The vertical axes is the deformation parameter c. On the left hand side, we see the bifurcation diagram of the function $f(x) = x^6 - x^4 + cx^2$, on the right hand side the bifurcation diagram of the function $f(x) = x^5 - x^4 + cx^2$. As done in class and homework, the bolder continuously drawn graphs show the motion of the local minima and the lighter dotted lines show the motion of the local maxima. In both cases, determine the catastrophe for the criticaal point $x = 0$.





Problem 13) Applications (10 points)

The **Laplace distribution** is a distribution on the entire real line which has the probability density $f(x) = e^{-|x|}/2$. The variance of this distribution is the integral

$$\int_{-\infty}^{\infty} x^2 f(x)\, dx \ .$$

Find it.

Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

5/7/2021: Final Practice F

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 9 AM on May 8th. Do not communicate with anybody related to the class during the exam period and with nobody at all about the exam.

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5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points) No justifications are needed.

1) T F If a function $f(x)$ has a critical point 0 and $f''(0) = 0$ then 0 is neither a maximum nor minimum.

2) T F If $f' = g$ then $\int_0^x g(x) = f(x)$.

3) T F The function $f(x) = 1/x$ has the derivative $\log(x)$.

4) T F The function $f(x) = \arctan(x)$ has the derivative $1/\cos^2(x)$.

5) T F The fundamental theorem of calculus implies that $\int_a^b f'(x)\ dx = f(b) - f(a)$.

6) T F $\lim_{x\to 8} 1/(x-8) = \infty$ implies $\lim_{x\to 3} 1/(x-3) = \omega$.

7) T F A continuous function which satisfies $\lim_{x\to -\infty} f(x) = 3$ and $\lim_{x\to\infty} f(x) = 5$ has a root.

8) T F The function $f(x) = (x^7 - 1)/(x - 1)$ has a limit at $x = 1$.

9) T F If $f_c(x)$ is an even function with parameter c and $f'(0) = 0$ and for $c < 3$ the function is concave up at $x = 0$ and for $c > 3$ the function is concave down at $x = 0$, then $c = 3$ is a catastrophe.

10) T F The function $f(x) = +\sqrt{x^2}$ has a continuous derivative 1 everywhere.

11) T F A rower rows on the Charles river leaving at 5 PM at the Harvard boat house and returning at 6 PM. If $f(t)$ is the distance of the rower at time t to the boat house, then there is a point where $f'(t) = 0$.

12) T F A global maximum of a function $f(x)$ on the interval $[0, 1]$ is a critical point.

13) T F A continuous function on the interval $[2, 3]$ has a global maximum and global minimum.

14) T F The intermediate value theorem assures that if f is continuous on $[a, b]$ then there is a root of f in (a, b).

15) T F On an arbitrary floor, a square table can be turned so that it does not wobble any more.

16) T F The derivative of $\log(x)$ is $1/x$.

17) T F If f is the marginal cost and $F = \int_0^x f(x)\ dx$ the total cost and $g(x) = F(x)/x$ the average cost, then points where $f = g$ are called "break even points".

18) T F At a function party, Log talks to Tan and the couple Sin and Cos, when she sees her friend Exp alone in a corner. Log: "What's wrong?" Exp: "I feel so lonely!" Log: "Go integrate yourself!" Exp sobbs: "Won't change anything." Log: "You are so right".

19) T F If a car's position at time t is $f(t) = t^3 - t$, then its acceleration at $t = 1$ is 6.

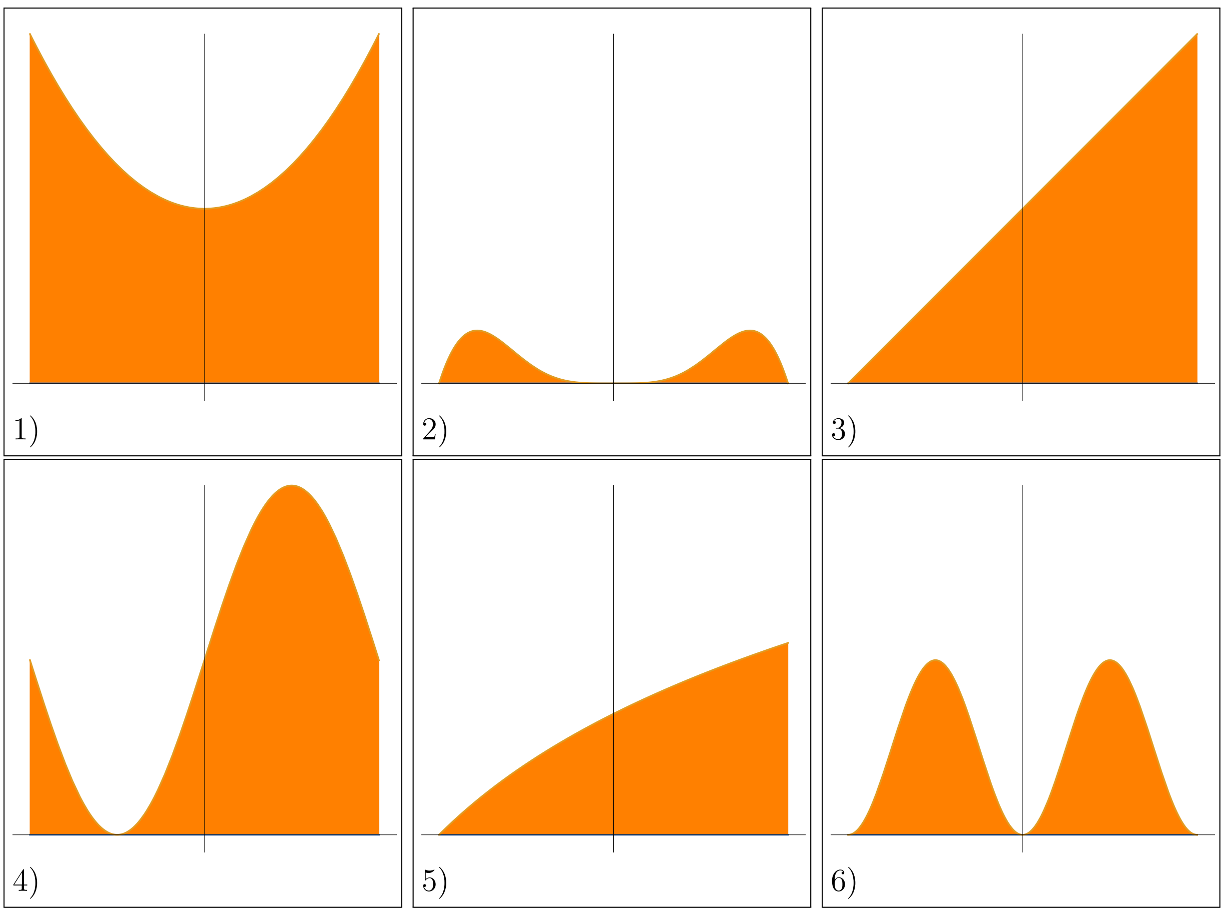
20) | T | F | For trig substitution, the identities $u = \tan(x/2)$, $dx = \frac{2du}{(1+u^2)}$, $\sin(x) = \frac{2u}{1+u^2}$, $\cos(x) = \frac{1-u^2}{1+u^2}$ are useful.

Problem 2) Matching problem (10 points) No justifications are needed.

a) Match the following integrals with the graphs and (possibly signed) areas.

Integral	Enter 1-6
$\int_{-1}^{1} \sin(\pi x) x^3 \, dx$.	
$\int_{-1}^{1} \log(x+2) \, dx$.	
$\int_{-1}^{1} x + 1 \, dx$.	

Integral	Enter 1-6
$\int_{-1}^{1} (1 + \sin(\pi x)) \, dx$.	
$\int_{-1}^{1} \sin^2(x) \, dx$.	
$\int_{-1}^{1} x^2 + 1 \, dx$.	



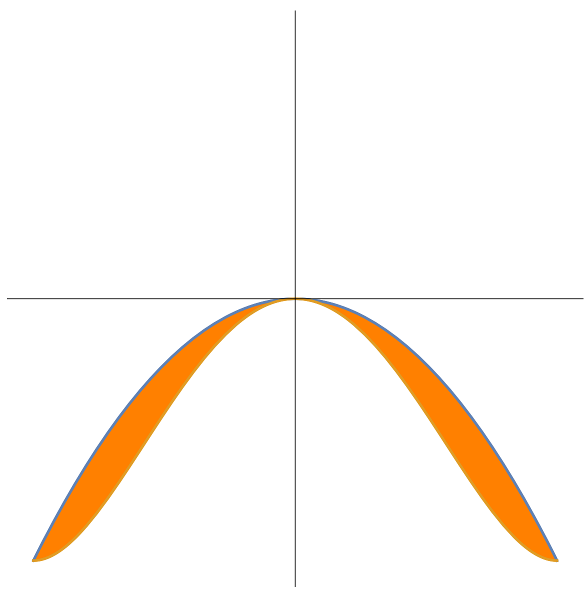
Problem 3) Matching problem (10 points) No justifications are needed.

Determine from each of the following functions, whether discontinuities appears at $x = 0$ and if, which of the three type of discontinuities it is at 0.

Function	Jump discontinuity	Infinity	Oscillation	No discontinuity
$f(x) = \log(\|x\|^5)$				
$f(x) = \cos(5/x))$				
$f(x) = \cot(1/x)$				
$f(x) = \sin(x^2)/x^3$				
$f(x) = \arctan(\tan(x - \pi/2))$				
$f(x) = 1/\tan(x)$				
$f(x) = 1/\sin(x)$				
$f(x) = 1/\sin(1/x)$				
$f(x) = \sin(\exp(x))/\cos(x)$				
$f(x) = 1/\log\|x\|$				

Problem 4) Area computation (10 points)

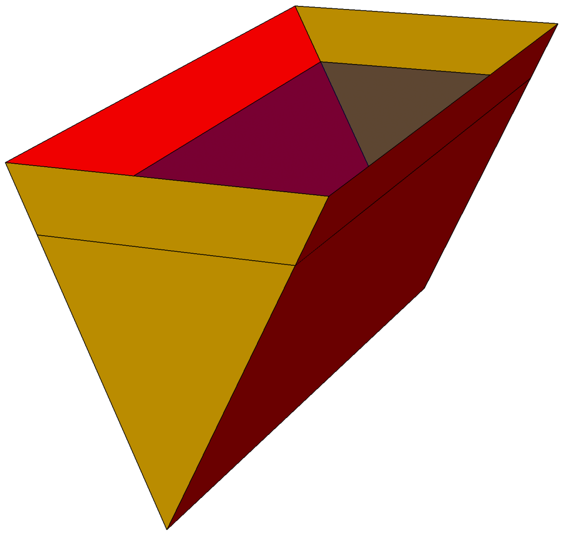
Find the area of the region enclosed by the graphs of the function $f(x) = x^4 - 2x^2$ and the function $g(x) = -x^2$.



Problem 5) Volume computation (10 points)

A farmer builds a bath tub for his warthog "Tuk". The bath has triangular shape of length 10 for which the width is $2z$ at height z. so that when filled with height z the surface area of the water is $20z$. If the bath has height 1, what is its volume?

P.S. Don't ask how comfortable it is to soak in a bath tub with that geometry. The answer most likely would be "Noink Muink".



Problem 6) Definite integrals (10 points)

Find the following definite integrals

a) (3 points) $\int_1^2 \sqrt{x} + x^2 - 1/\sqrt{x} + 1/x \; dx$.

b) (3 points) $\int_1^2 2x\sqrt{x^2 - 1} \; dx$

c) (4 points) $\int_1^2 2/(5x - 1) \; dx$

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

a) (3 points) $\int \frac{3}{1+x^2} + x^2 \; dx$

b) (3 points) $\int \frac{\tan^2(x)}{\cos^2(x)}\, dx$

c) (4 points) $\int \log(5x)\, dx$.

Problem 8) Chain rule (10 points)

A juice container of volume $V = \pi r^2 h$ changes radius r but keeps the height $h = 2$ fixed. Liquid leaves at a constant rate $V'(t) = -1$. At which rate does the radius of the bag shrink when $r = 1/2$? Differentiate $V(t) = \pi r(t)^2 h$ for the unknown function $r(t)$ and solve for $r'(t)$, then evaluate for $r = 1/2$.

Problem 9) Global extrema (10 points)

We build a chocolate box which has 4 cubical containers of dimension $x \times x \times h$. The total material is $f(x, h) = 4x^2 + 12xh$ and the volume is $4x^2h$. Assume the volume is 4, what geometry produces the minimal cost?

Problem 10) Integration techniques (10 points)

Which integration technique works? It is enough to get the right technique and give the first step, not do the actual integration:

a) (2 points) $\int (x^2 + x + 1)\sin(x)\, dx$.

b) (2 points) $\int x/(1 + x^2)\, dx$.

c) (2 points) $\int \sqrt{4 - x^2}\, dx$.

d) (2 points) $\int \sin(\log(x))/x$.

e) (2 points) $\int \frac{1}{(x-6)(x-7)}\, dx$.

Problem 11) Hopital's rule (10 points)

Find the following limits as $x \to 0$ or state that the limit does not exist.

a) (2 points) $\frac{\tan(x)}{x}$

b) (2 points) $\frac{x}{\cos(x)-x}$.

c) (2 points) $x\log(1+x)/\sin(x)$.

d) (2 points) $x\log(x)$.

e) (2 points) $x/(1-\exp(x))$.

Problem 12) Applications (10 points)

The cumulative distribution function on $[0,1]$

$$F(x) = \frac{2}{\pi}\arcsin(\sqrt{x})$$

defines the **arc-sin** distribution.

a) Find the probability density function $f(x)$ on $[0,1]$.

b) Verify that $\int_0^1 f(x)\,dx = 1$.
Remark. The arc sin distribution is important chaos theory and probability theory.

Problem 13) Data (10 points)

Find the best linear fit $y = mx$ through the data points $(3,5),(1,1),(-1,1),(2,2)$.