Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2024

5/8/2024: Final Exam Practice A

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

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14		10
Total:		140

Problem 1) TF questions (10 points) No justifications are needed.

1) T **F** $e^{(x^y)} = e^{(y^x)}$ for all real numbers x, y.

Solution:
x^y is not the same than y^x.

2) **T** F There was a time in you life when your age $t \in \mathbb{R}$ in units of years was exactly your height $h \in \mathbb{R}$ in units in meters.

Solution:
This follows from the intermediate value theorem. At $t = 0$ you have positive length and zero age and now you have an age in years which is bigger than your height in meters.

3) T **F** The function $f(x) = \sin(x)/x$ is called the entropy function.

Solution:
It is the sinc function. This function appears in the fundamental theorem of trigonometry.

4) **T** F Applying the Newton step with the function $f(x) = x^2 - 2$ at the point $x = 1$ gives $T(x) = 1 + 1/2 = 3/2$.

Solution:
yes

5) **T** F The family of functions $f_c(x) = c(x-1)^2 + 3$ experiences a catastrophe at $c = 0$.

Solution:
It is indeed at c=0

6) T F If $F(x) = x^2 + 3x$ then $f(x) = 2x + 3$ is called marginal cost.

Solution:
Yes.

7) T F $\sin(3\pi/2) = -1$.

Solution:
Yes,

8) T F The cotangent function is monotonically decreasing on the open interval $(\pi/4, \pi/2)$.

Solution:
Indeed, its derivative is $1/\cos^2(x)$.

9) T F If F is a CDF, then $F(0) = 0$.

Solution:
It depends. F(0)=0 only holds if the density $f(x)$ is zero for negative x.

10) T F If f is a PDF, then $\int_{-\infty}^{\infty} x^2 f(x)\, dx$ is called the variance of the distribution.

Solution:
It is $\int_{-\infty}^{\infty} (x - m)^2 f(x)\, dx$ where m is the mean.

Problem 2) Algebra (10 points)

a) (5 points) Which of the following expressions are integers?

Expression	Integer (give value) or not?
$\arccos(\cos(2))$	
$e^{\ln(e^2)}$	
$\tan(\arctan(\tan(\pi/4)))$	
$e^{(e^0)}$	
$\ln(\ln(e))$	

Solution:
a) yes, 2
b) no e^2
c) yes, 1
d) no, e^1
e) yes, 0

b) (5 points) Which of the following expressions are true for all x?

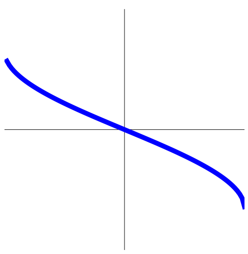
Expression	True or False
$\cos^2(2x) + \sin^2(2x) = 1$	
$\cos^2(2x) - \sin^2(2x) = \cos(2x)$	
$e^{e^{\ln(\ln(x))}} = x$	
$\sin(\arcsin(x)) = x$	
$\tan(\arctan(x)) = x$	

Solution:
True, False, True, False, True

Problem 3) Functions (10 points)

Match the functions names with their graphs (1-4) their derivatives (A-D) (middle row) and second derivatives (a-d) (last row).

Function	fill in 1)-4)	fill in A)-D)	fill in a)-d)
$\sin(x)/x$			
$\tan(x)$			
$-\arcsin(x)$			
$1/(1+x^2)$			



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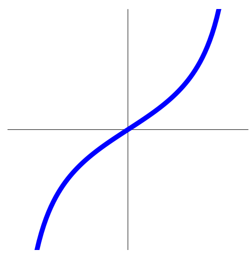

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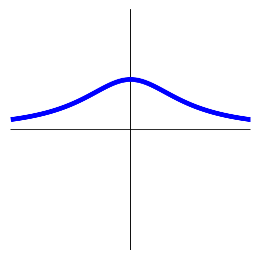

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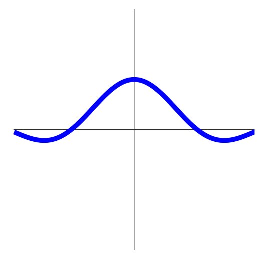

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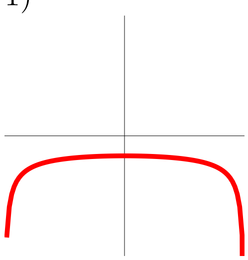
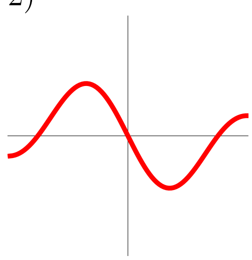


A)



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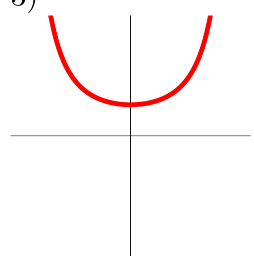
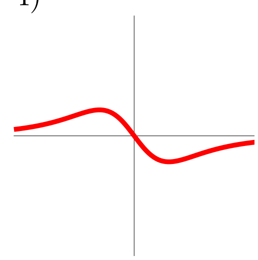


C)



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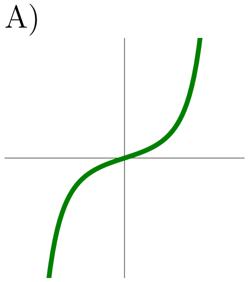
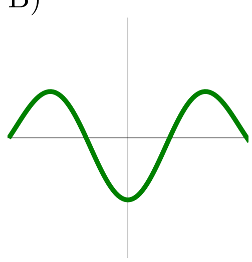


a)



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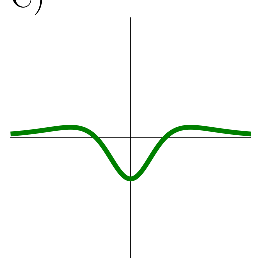

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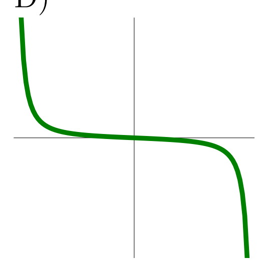


d)

Solution:
4,B,b
2,C,a
1,A,d
3,D,c

Problem 4) Limits, Continuity (10 points)

a) (5 points) Which of the following limits exists in the limit $x \to 0$.

Function	exists	does not exist
$\sin^4(x)/x^4$		
$\arctan(x)/x$		
$\ln|x|/(x-1)$		
$\cos(x)/(x-1)$		
$(x^{10}-1)/(x-1)$		

Solution:
b) Every limit exists except the case $\ln|x|/(x-1)$.

b) (5 points) Describe in terms of the concept of "limit" what it means that a function $f(x)$

is continuous at 1	
is differentiable at 1	
has a jump discontinuity at 1	

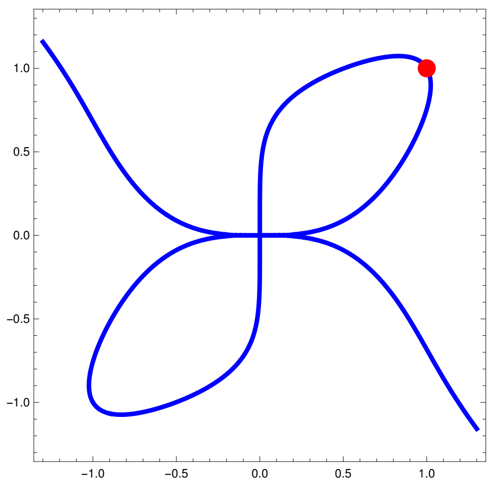
Solution:
- The limit $\lim_{x \to 1} f(x)$ exists.
- The limit $\lim_{h \to 0}[f(1+h) - f(1)]/h$ exists.
- The left and right limit at the point 1 do both exist but are not the same. $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$. $f(x) = \text{sign}(x-1)$ is an example, where the left limit is -1 and the right limit is 1.

Problem 5) Related Rates (10 points)

The coordinates of two cars on a freeway intersection are $x = x(t)$ and $y = y(t)$. We know

$$x^7 + y^7 - 2xy^2 = 0 \ .$$

Assume we know $x'(0) = 3$ and $x(0) = 1, y(0) = 1$. Find the derivative $y'(t)$.



Solution:
Differentiate the relation with respect to t and solve for y':

$$7x^6x' + 7y^6y' - 2x'y^2 - 4xyy' = 0 \ .$$

Therefore, we can solve for

$$y' = (7x^6x' - 7y^6x')/(4xy - 6y^2) \ .$$

The final answer is $\boxed{-5}$.

Problem 6) Integrals (10 points)

Find the anti-derivatives of the following functions:

a) (2 points) $f(x) = x^7 - \frac{1}{x}$

b) (2 points) $f(x) = x\ln(x) + \frac{1}{1+x^2}$.

c) (2 points) $f(x) = \frac{2x}{x^2+1} + \frac{1}{x^2-4}$.

d) (2 points) $f(x) = \sqrt{16 - x^2} + \frac{1}{\sqrt{1-x^2}}$.

e) (2 points) $f(x) = \ln(x) + \frac{1}{x\ln(x)}$.

Solution:
a) $x^8/8 - \ln(x) + C$.
b) $x^2\ln(x)/2 - x^2/4 + \arctan(x) + C$.
c) $\ln(x^2 + 1) - \ln(x - 2)/4 + \ln(x + 2)/4 + C$.
d) $\arcsin(x/4) + (1 - \cos(2\arcsin(x/4)))/4 + C$.
e) $x\ln(x) - x + \ln(\ln(x)) + C$.

Problem 7) Extrema (10 points)

A **trapezoid** of height $2x$, bottom length x and top length $(4-2x^2)$ has area

$$f(x) = x(x + (4-2x^2)) = 4x + x^2 - 2x^3 .$$

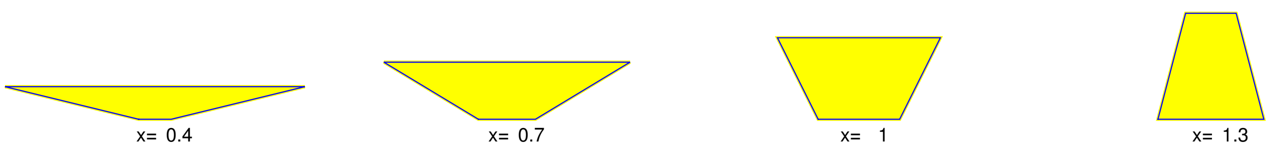
The figure below illustrates the situation for some x values.

a) (3 points) For which x does f have a local maximum? Use the second derivative test to check.

b) (3 points) On which interval $[a, b]$ does the problem make sense (height, lengths and area can not be negative)? What are the function values on the boundary?

c) (2 points) There a theorem which assures that $f(x)$ has a global maximum on that interval $[a, b]$? What is the name of the theorem?

d) (2 points) What is the global minimum of $f(x)$ on the interval of consideration?



Solution:
$f'(x) = 4 + 2x - 6x^2$ for $x = 1$ and $x = -2/3$. Only $x = 1$ makes sense in this set-up. The function is 0 at $x = 0$ and $x = (1+\sqrt{33})/4$ but the function is only defined on $[0, \sqrt{2}]$, because $4 - 2x^2$ has to be positive. The minimum is at $x = 0$. The point $x = 1$ is a maximum because $f''(x) = 2 - 12x$ is -10 at $x = 1$. The extremal value theorem assures that we have a maximum.

Problem 8) Substitution (10 points)

a) (5 points)

$$\int \cos(\cos(\cos(x)))\sin(\cos(x))\sin(x)\ dx\ .$$

b) (5 points)

$$\int \frac{2\ln(\ln(\ln(x)))}{\ln(\ln(x))\ln(x)x} dx\ .$$

Solution:
a) $\sin\cos(\cos(x))) + C$
b) $\ln^2(\ln(\ln(x))) + C$.

Problem 9) Integration by parts (10 points)

a) (5 points) Compute the following anti-derivative:

$$\int (x-2)^4 e^{x/2} \, dx \; .$$

b) (5 points) And now have some fun riding the "merry go round" for the following integral

$$\int e^x \sin(x) \, dx \; .$$

Solution:
a) Use the Tic-Tac-Toe method.

$(x-2)^4$	$\exp(x/2)$	
$4(x-2)^3$	$2\exp(x/2)$	$\oplus$
$12(x-2)^2$	$4\exp(x/2)$	$\ominus$
$24(x-2)$	$8\exp(x/2)$	$\oplus$
24	$16\exp(x/2)$	$\ominus$
0	$32\exp(x/2)$	$\oplus$

and add up.
b) $(e^x \sin(x) - e^x \cos(x))/2 + C$.

Problem 10) Fractions (10 points)

a) Integrate the definite integral $\int_5^6 \frac{4}{(x-3)(x+1)}\, dx$.

b) Integrate

$$\int \frac{1}{(x+8)(x+4)(x+2)}\, dx \ .$$

Solution:
a) Write the function as $-1/(x-3) + 1/(x+1)$ and integrate $\ln(x-3) + \ln(x+1)$. The definite integral is $\ln(3) + \ln(7) - \ln(2) - \ln(6)$.
b) Use the residue method to rewrite the integrand as $\ln(x+8)/24 - \ln(x+4)/8 + \ln(x+2)/12$ Then integrate.

Problem 11) Applications I (10 points)

a) (2 points) Find the CDF $\int_0^x f(t)\,dt$ for the PDF which is $f(x) = \exp(-x/3)/3$ for $x \geq 0$ and 0 for $x < 0$.

b) (2 points) Perform a single Newton step for the function $f(x) = \sin(x)$ starting at $x = \pi/3$.

c) (2 points) Check whether the function $f(x) = 1/(2x^2)$ on the real line $(-\infty, \infty)$ is a probability density function.

d) (2 points) A rower produces the power $P(t)$ is $\sin^2(10t)$. Find the energy $\int_0^{2\pi} P(t)\,dt$ when rowing starting at time $t = 0$ and ending at $t = 2\pi$.

e) (2 points) What is the frequency of the Midi number 10 if you recall the formula $f = 4402^{(s-69)/12}$?

Solution:

a) The anti-derivative is $\boxed{(1 - e^{-x/3}}$.

b) $\pi/3 - \tan(\pi/3) = \boxed{\pi/3 - \sqrt{3}}$.

c) The integral does not exist. It is **not** a probability density function.

d) Integrate $\int_0^{2\pi} \sin^2(10t)\,dt = \int_0^{2\pi}(1 - \cos(20t))/2\,dx = x/2 - \sin(20t)/40|_0^{2\pi} = \boxed{\pi}$.

e) $f = 4402^{(10-69)/12}$.

Problem 12) Applications II (10 points)

Lets look at a least square problem. Given the data points $(1, 2), (2, 3), (-3, -5)$, find the line $y = mx$ which best fits the data? You need to minimize

$$\sum_k (mx_k - y_k)^2$$

if (x_k, y_k) stands for the data points. Use one of the derivative theorems to verify that you got a minimum.

Solution:
Write it out $f(m) = (m - 2)^2 + (2m - 3)^2 + (-3m + 5)^2$. Now differentiate $2(m - 2) + 4(2m - 3) + (-6)(-3m + 5) = 0$ which simplifies to $28x - 46$. We get $46/28$.

Problem 13) Definitions (10 points)

a) (2 points) What is the average rate of change of $f(x) = \sin(x)$ on the interval $[0, \pi/2]$.

b) (2 points) Why is the sigmoid function called logistic function ?

c) (2 points) If m is the mean of a probability distribution, then what is the name for $\int_{-\infty}^{\infty} (x - m)^2 f(x)\, dx$.

d) (2 points) What do we call the points for which $f(x) = 0$?

e) (2 points) What do we call the points for which $f'(x) = 0$?

Solution:
a) $2/\pi$.
b) Because it is a solution of the logistic equation.
c) Variance
d) Roots
e) Critical points

Problem 14) Theorems (10 points)

The statement	is called the
A continuous function takes a maximum on $[0,1]$	.
$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$	.
$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$	.
$\int_0^x f'(t)\, dt = f(x) - f(0)$	.
A wobbling table can be fixed by turning	
Break-even points are minima of the average cost	
If $f(0) = f(1) = 5$ there is $0 < x < 1$ with $f'(x) = 0$	
If $f(0) = 5,\ f(1) = 8$, there is $0 < x < 1$ with $f'(x) = 3$	
All functions can be approximated by neural nets	
If $f(0) = 4$ and $f(1) = -4$, then there is $0 < x < 1$ with $f(x) = 0$	

Solution:
Extremal value theorem
Fundamental theorem of trig
Hospital's rule
Fundamental theorem of calculus
Wobbly table theorem
Strawberry theorem
Rolle's theorem
Mean value theorem
Universal approximation theorem
Intermediate value theorem