INTRODUCTION TO CALCULUS

MATH 1A

Unit 5: Continuity

5.1. A **function** f is called **continuous** at a point x_0 if a value $f(x_0)$ can be found such that $f(x) \to f(x_0)$ for $x \to x_0$. A function f is **continuous on the interval** $[a, b]$ if it is continuous for every point x in the interval $[a, b]$. This means intuitively, we can **draw the graph of the function without lifting the pencil**. In (a, b), the limit needs to exist both from the right and from the left. Continuity means that small changes in x results in small changes of $f(x)$. Any polynomial like $x^3 + 2x - 4$ or trig functions like $\cos(x), \sin(x)$ or exponential functions $\exp(x)$ are continuous.

Rules:
a) If f and g are continuous, then $f + g$ is continuous.
b) If f and g are continuous, then $f * g$ is continuous.
c) If f and g are continuous and if $g \neq 0$ everywhere, then f/g is continuous.
d) If f and g are continuous, then $f \circ g(x) = f(g(x))$ is continuous.

5.2. The **squeeze theorem** is a tool to check continuity at a point a.

If $g(x) \leq f(x) \leq h(x)$ for functions g, h continuous at a and $g(a) = h(a) = b$, then f is continuous at a and $f(a) = b$.

The reason is that $|h(x) - g(x)| \to 0$ and $g(x) \to b$ and $|f(x) - b| \leq |h(x) - g(x)| + |g(x) - b| \leq |h(x) - g(x)| + |g(x) - b| \to 0$ so that also $f(x) \to b$.

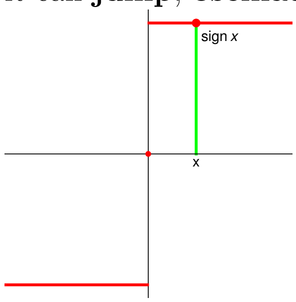
5.3. The function $f(x) = 1/x$ is continuous except at $x = 0$. There is **pole discontinuity** at $x = 0$. The graph has a **vertical asymptote**.

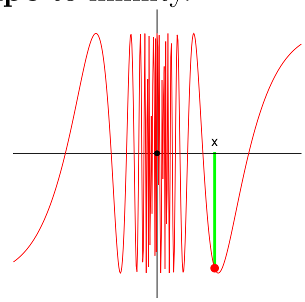
5.4. The logarithm function $f(x) = \ln|x|$ is continuous for all $x \neq 0$. It can not be fixed $x = 0$ because $f(x) \to -\infty$ for $|x| \to 0$.

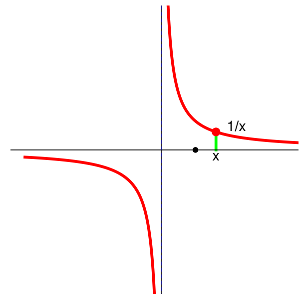
5.5. The **co-secant function** $\csc(x) = 1/\sin(x)$ is not continuous at $x = 0, x = \pi$ and any multiple of π.

5.6. The function $f(x) = \sin(\pi/x)$ is continuous everywhere except at $x = 0$. It fails continuity because of **oscillation**. We can approach $x = 0$ in ways that $f(x_n) = 1$ and such that $f(z_n) = -1$. Just pick $x_n = 2/(4k + 1)$ or $z_n = 2/(4k - 1)$.

5.7. There are three major reasons, why a function can be not continuous at a point: it can **jump**, **oscillate** or **escape** to infinity.







Homework, Due Friday 2/2/2024

Problem 5.1: a) Define $f(x) = x^2 \cos(1/x)$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$. Use the squeeze theorem to see that this function is continuous everywhere.
b) Define $f(x) = \cos(1/x)$ for $x \neq 0$ and $f(x) = 1$ for $x = 0$. Verify that this function is not continuous at 0.

Solution:
a) The only problem is $x = 0$. Lets look at a point x near 0 but not equal to 0. The value of $|\cos(1/x)|$ is ≤ 1. So, $|x^2 \cos(1/x)| \leq x^2$. We can use the squeeze theorem for $g(x) = -x^2$ and $h(x) = x^2$.
b) The function takes values -1 and 1 arbitrarily close to 0.

Problem 5.2: The number of users (in millions) of a social network is modeled as a function that is linear $U(t) = at + b$ for $t \geq 5$ and exponential $U(t) = ce^{kt}$ for $t \in [0, 5]$. Assume that we have 2 million users initially $U(0) = 2$. Data fitting leads to $a = 220$ and $b = 380$ for the linear growth. Determine k so that the function is continuous at 5.

Solution:
The right limit at $t = 5$ is $b = 220 * 5 + 380 = 1480$. We have $U(0) = 2 = c$ so that $c = 2$. Now fix $2e^{k5} = 1480$. This means $k = \ln(740)/5$.

Problem 5.3: a) The function $f(x) = (e^{2x} - 1)/(e^x - 1)$ is not defined at $x = 0$. Can you find a value $f(0) = b$ so that the function is continuous everywhere?
b) The function $f(x) = (x^3 + 2x^2 - 2x - 1)/(x - 1)$ is not defined at $x = 1$. Can you find a value $f(1) = b$ so that with this postulate the function is continuous?

Solution:
a) We can factor and see that for $x \neq 1$ the function is $f(x) = e^x + 1$. There is therefore a limit $b = 2$.
b) We can factor and see that for $x \neq 1$ we can factor out $(x-1)$ and have $1 + 3x + x^2$. There is therefore a limit $b = 5$.

Problem 5.4: Which of the following functions are continuous everywhere?
a) $f(x) = \text{sign}(x) + \sin(1/x)$
b) $f(x) = x\text{sign}(x) + \sin(1/x)$
c) $f(x) = \text{sign}(x) + x\sin(1/x)$
d) $f(x) = x\text{sign}(x) + x\sin(1/x)$

Solution:
Only d) is continuous. In b),c) two functions are not continuous at $x = 0$ and the other are so that we have no continuity. in a), we check that we have values 2 arbitrarily close to 0 and in -2 arbitrarily close to 0.

Problem 5.5: Which functions can be made continuous by "fixing broken places" (assign a value to an initially not defined point)? If the function can be fixed show how.
a) $\text{sinc}(5x) + \sin(x)/(2 + \sin^2(x)) + (x^3 - 1)/(x - 1)$, b) $\sin(\tan(x))$
c) $\tan(\sin(x)) + \frac{x^2+5x+x^4}{x-3}$
d) $\tan(2\sin(x)) + \frac{x^2+5x+x^4}{x-3}$

Solution:
a) can be fixed. b) can not be fixed at $x = \pi/2$. c) can not be fixed The first function is continuous but the second still was not fixed.
d) can not be fixed. Again, the second term has a pole. But also the first term becomes problematic because $2\sin(x)$ which is fed into the tangent function reaches the point $\pi/2 \sim 1.5$ and so produces a crazy oscillation.

Oliver Knill, knill@math.harvard.edu, Math 1a, Spring, 2024