**INTRODUCTION TO CALCULUS**

MATH 1A

# Unit 6: Derivative

**6.1.** The **derivative** of a function $f(x)$ at a point $x$ is defined as the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if the limit exists. It is the **instantaneous rate of change** we introduced earlier: it is the slope of the tangent at the point $x$.

**6.2.** Let us look at the function $f(x) = x^2$. We have

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{2hx + h^2}{h} \; .$$

For $h \neq 0$, we can divide by $h$ and equate this to $2x + h$. We can now take the limit $h \to 0$ and see that $f'(x) = 2x$.

**6.3.** We will derive in class that in general, the function $f(x) = x^n$ leads to

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^n - x^n}{h} = \frac{nhx + \dots}{h}$$

which similarly as before simplifies to $nx^{n-1} + h(R(x))$, where $R(x)$ is a polynomial. Now again, for $h \to 0$, have

$$\boxed{\tfrac{d}{dx} x^n = nx^{n-1}}$$

**6.4.** For the **exponential function** $f(x) = e^x$, one can also compute the derivative

$$\frac{e^{x+h} - e^x}{h} = \frac{[e^x e^h - e^x]}{h} = e^x \frac{e^h - 1}{h} \; .$$

Now we can see that the limit $(e^h - 1)/h$ goes to 1 as $h \to 0$. Proving this depends on how the exponential function is defined. A calculator for example implements the exponential function as $e^h = 1 + h + h^2/2 + h^3/6 + \cdots$ from which you can see that $e^h - 1$ is divisible by $h$. Taking the limit $h \to 0$ gives then 1.

$$\boxed{\tfrac{d}{dx} e^x = e^x}$$

## Homework

This homework is due on Monday 2/05/2024.

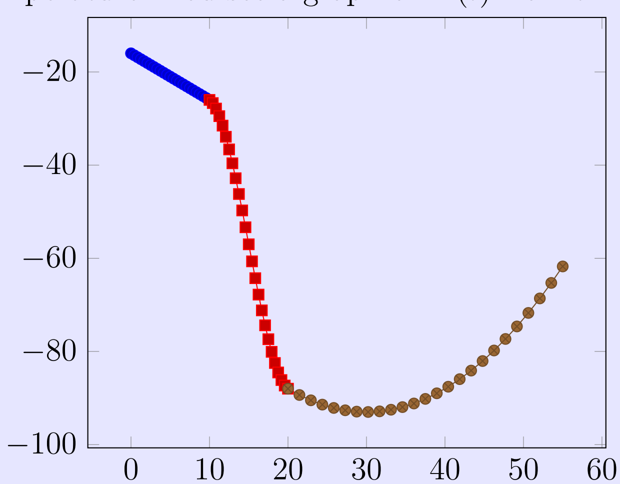
**Problem 6.1:** a) Find the limit $\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$ at the point $x = 2$.
b) Do the same with the function $x^3 - x$ instead of $x^3$.

**Problem 6.2:** a) Use the definition of the derivative to compute the derivative of $(x+1)/(x-2)$ at $x = 0$.
b) Compute the limit $\lim_{h \to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h}$.

**Problem 6.3:** Give in each case a function with the property or explain why it does not exist:
a) $f$ is continuous at $x = 3$ but not differentiable at $x = 3$.
b) $f$ is continuous at $x = 3$ and differentiable at $x = 3$.
c) $f$ is differentiable at $x = 3$ but not continuous at $x = 3$.

**Problem 6.4:** Find the limit $f'(x) = \lim_{h \to 0}[f(x+h) - f(x)]/h$. for $f(x) = 1/x$.

**Problem 6.5:** In this QRD problem we want to see what $f'$ tells about $f$. The water level $L(t)$ of the **Aral sea** dropped between 1960 and 1970 by 21 cm/year, then from 1970-1980, it decreased by 57 cm per year. Afterwards until 2015 (t=50) the drop in water level started accelerating, due to positive feedback between evaporation and Sea Surface Temperature. You see a graph of $L'(t)$ from $t = 0$ to $t = 55$.




Assume that the water level was 54 m in 1960 (t=0), draw a qualitative picture on how $L(t)$ looks like.

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