INTRODUCTION TO CALCULUS

MATH 1A

Unit 7: Basic Derivatives

7.1. We have already seen the key power identity:

$$\frac{d}{dx}x^n = nx^{n-1} \, .$$

7.2. As a general rule we can state already now is that for any constant c

$$\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$$

In short $(cf)' = cf'$. The reason is that for every h, we have the property that the average rate of change $(f(x+h) - f(x))/h$ has the property that **it plays nice with linearity**. This property goes over to the limit when the Planck constant h goes to zero.

7.3. An other general rule is the **addition rule**:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

7.4. Now we can compute the derivative of **any polynomial**: for example: lets compute the derivative of $4x^3 + 3x^2 + x + 3$. The answer is $12x^2 + 6x + 1$.

7.5. We have defined $f(x) = e^x$ as the compound interest limit $(1+h)^{x/h}$ for $h \to 0$. [1] Since we can check that $(f(x+h) - f(x))/h = f(x)$ for any $h > 0$, the exponential function also has the property $\frac{d}{dx}e^x = e^x$ and more generally

$$\frac{d}{dx}e^{cx} = ce^{cx}.$$

[1] This limit e^x exists: use the squeeze theorem for $g(x) \leq f(x) \leq h(x)$ with the decreasing $g(h) = (1+h)^{x/h}$ and increasing $k(h) = (1+h)^{x/h}(1+h)$ using that for $h \leq 1$ one has $k(h) - g(h) = h(1+h)^{x/h} \leq h2^x$ which converges to 0 for $h \to 0$.

7.6. If you increase x by a factor c faster, also the slope gets scaled by c. In general

$$f'(cx) = cf(cx)$$

You generalize this slightly in the homework. Later we will learn it as a special case of the chain rule.

7.7. In order to see the derivatives of the trig functions, remember first the **fundamental theorem of trigonometry** and companion identity:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \ , \lim_{x\to 0} \frac{\cos(x)-1}{x} = 0 \ .$$

Both these limits follow from the squeeze theorem. (See the short video). If we divide one of the addition formulas for trig functions by h

$$\sin(x+h) - \sin(x) = \cos(x)\sin(h) + \sin(x)(\cos(h)-1)$$

we get $\cos(x)$ in the limit $h \to 0$. If we divide the second addition formula

$$\cos(x+h) - \cos(x) = \cos(x)(\cos(h)-1) - \sin(x)\sin(h)$$

by h and take the limit we get $-\sin(x)$. We have shown

$$\frac{d}{dx}\sin(x) = \cos(x), \ \frac{d}{dx}\cos(x) = -\sin(x)$$

7.8. In the homework for today you have shown from the definition:

$$\lim_{h\to 0}[\frac{1}{x+h} - \frac{1}{x}]/h = -\frac{1}{x^2} \ .$$

You have also seen in the homework that

$$\lim_{h\to 0}[\sqrt{x+h} - \sqrt{x}]/h = \frac{1}{2\sqrt{x}} \ .$$

These are special cases for the following formula for x^n. Indeed, for **any real number** n also negative ones, we have

$$\frac{d}{dx}x^n = nx^{n-1}.$$

7.9. By the way, we can see this also from writing $x^n = e^{n\ln(x)}$, using the chain rule (covered later in the course) and using $\frac{d}{dx}\ln(x) = 1/x$ which we are going to look at next.

7.10. In the last worksheet, we simplified $\lim_{h\to 0}\frac{\ln(x+h)-\ln(x)}{h}$ to $\lim_{h\to 0}\frac{\ln(1+h/x)}{h}$. We need to get the limit $\ln(1+h)/h$. Since the limit $h \to 0$ of $(1+h)^{1/h}$ gives e by definition, $\ln(1+h)/h \to 1$ and so $\ln(1+h/x)/h \to 1/x$.

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

Homework

This homework is due on Wednesay 2/7, 2024.

Problem 7.1: Compute the following derivatives.
a) $2x^5 + 3x^2 + 4x + 8$.
b) $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$.
c) $1 + x + x^2/2 + x^3/6 + x^4/24 + x^5/120$
d) $\sqrt{x} + x^{3/2} + x^{5/2}$
e) $(1 + x)(1 + x + x^2 + x^3 + x^4)$

Problem 7.2: We have seen that $\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$.
a) Why is $\frac{d}{dx}\frac{1}{x-5} = -\frac{1}{(x-5)^2}$?
b) Formulate a general rule $\frac{d}{dx}f(x-a) = f'(..........)$ which holds for any differentiable function f.
c) Formulate a general rule $\frac{d}{dx}f(cx+b) = f'(.........)$ which holds for any differentiable function f.

Problem 7.3: Compute the following derivatives
a) $4\sin(3x) + 7\cos(9x)$
b) $\ln(x) + \ln(2x) + \ln(3x)$
c) $10e^{11x} + 8e^{20x} - 20e^{100x}$
d) $9x^2 + \frac{1}{x^7} + 2\cos(3x) + \ln(7x) - e^x$.
e) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$.

Problem 7.4: a) Compute the derivative of $f(x) = \pi^x$. You first might have to rewrite the function in a form which allows you to use rules you know.
b) What is the derivative of a^x in general if $a > 0$ is an arbitrary number?

Problem 7.5: a) Compute the derivative of $f(x) = \sqrt{3x+5}$ from the rules you know.
b) In order to appreciate what we have achieved, compute the limit

$$\lim_{h \to 0}[f(x+h) - f(x)]/h \, .$$

for the function $f(x) = \sqrt{3x+5}$ the old way as in PSet 6.

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