INTRODUCTION TO CALCULUS

MATH 1A

Unit 8: Derivative Rules

8.1. You have all already used linearity of the derivative. If we multiply a function by a constant c, then the average rate of change $(f(x+h)-f(x))/h$ also gets multiplied by c. We can pass to the limit and see

$$(cf)' = cf'$$

8.2. Also, if we take the sum of two functions $f+g$, this is a new function, whose derivative is the sum of the derivatives of f and g

$$(f+g)' = f' + g'$$

The two properties together show that the process of going from f to f' is linear.

8.3. The product rule for differentiation follows from the identity

$$f(x+h)g(x+h) - f(x)g(x) = [f(x+h)-f(x)] \cdot g(x+h) + f(x) \cdot [g(x+h)-g(x)] \ .$$

When dividing by h we get on the left hand side the average rate of change of fg on $[x, x+h]$ and on the right the average rate of change of f times $g(x+h)$ plus f times the average rate of change of g. For $h \to 0$, this is

$$(fg)' = f'g + fg'$$

Figure 1. The product rule. Leibniz.

8.4. The quotient rule allows to differentiate $f(x)/g(x)$ if $g(x) \neq 0$:

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{[g(x)f'(x) - f(x)g'(x)]}{g^2(x)}$$

"High d low take low d high. Cross the line and square the low." For example, we can see that $\tan'(x) = 1/\cos^2(x)$.

Homework, due 2/9/2024

Problem 7.1: Compute the following derivatives using the product rule:
a) $\frac{x^2}{2} - \frac{2}{x^2}$
b) $e^{5\ln(x)+\ln(2)}$ first rewrite this
c) $(t + \frac{1}{t})(t - \frac{1}{t})$. Use the product rule. Smarty pants solution only additional.
d) $\sin(x)\cos(x)$
e) $\ln(x)e^x \sin(x)$

Solution:
a) $x + 4/x$. If we insist on using the product rule, treat 1/2 as a constant and x^2 as the second function then $f'g + fg' = 0x^2 + (1/2)2x = x$.
b) This is actually $2x^5$. The result is $10x^4$.
c) $(1 - 1/t^2)(t - 1/t) + (t + 1/t)(1 - 1/t^2) = 2t - 2/t$. (This could also be done by first simplifying $f(t) = t^2 - 1/t^2$ but we asked you to use the product rule).
d) $\cos^2(x) - \sin^2(x)$.
e) $1/x(e^x \sin(x)) + \ln(x)[e^x \sin(x) - e^x \cos(x)]$

Problem 7.2: Now compute the following derivatives using the quotient rule:
a) $\cot(x)$
b) $\frac{\ln(x)}{\ln(2x)}$
c) $\frac{x^2+2x+1}{x\sqrt{x}}$
d) $\frac{1+x}{1-x}$
e) $\frac{x^2-1}{x-1}$ Use the quotient rule. Smarty pants solution only additional.

Solution:
a) $-1/\sin^2(x)$.
b) $(\ln(2x)/x - \ln(x)/x]/(\ln(2x))^2$.
This could be simplified to $\ln(2)/(x(\ln(2x))^2$.
c) $2x + 2 - \frac{3}{2x^{5/2}}$.
d) $2/(x-1)^2$.
e) 1.

Problem 7.3: Compute the first two derivatives of the cotangent function by hand. This involves both the product and quotient rule. For c)-e) you can make use of computer assistance, if you like. a) $\cot'(x)$, b) $\cot''(x)$, c) $\cot'''(x)$, d) $\cot''''(x)$, e) $\cot'''''(x)$.

Solution:
a) $-1/\sin^2(x)$. (we had already done that!)
b) $\sin^2(x) * 0 + 12\sin(x)\cos(x)/\sin^4(x) = 2\cos(x)/\sin^3(x))$. This could also be written as $2\cot(x)\csc^2(x)$.
c) Here is what Mathematica gives $-2\csc^4(x) - 4\cot^2(x)\csc^2(x)$.
d) $8\cot^3(x)\csc^2(x) + 16\cot(x)\csc^4(x)$. This would be quite challenging to get by hand already.

Problem 7.4: We break here the Guinness record of the **most sophisticated differentiation problem** ever posed in a college calculus course. First define $f_0(x) = x$, then $f_1(x) = 1/(1+x), f_2(x) = 1/(1+1/(1+x))$ with the rule $f_n(x) = 1/(1 + f_{n-1}(x))$. We ask you to compute the derivative of f_{50}! Hint: We actually have $f_n(x) = \frac{F(n)+F(n-1)x}{F(n+1)+F(n)x}$, where $F(n)$ is the n'th Fibonacci number. We give you $F(50) = 12586269025, F(49) = 7778742049$.

Solution:
We differentiate $F(n) + F(n-1)x/(F(n+1) + F(n)x)$ using the quotient rule. It gives $[F(n-1)(F(n+1) + F(n)) - F(n)(F(n) + F(n-1))x]/(F(n+1) + F(n)x)^2$. This is the answer after plugging in $n = 50$:

$$\frac{F(49)(F(51) + F(50)) - F(50)(F(50) + F(49))x}{(F(51) + F(50)x)^2} .$$

Just state that $F(49), F(50)$ have been given and that $F(51) = F(49) + F(50) = 12586269025 + 7778742049$.

Problem 7.5: We test ChatGPT and ask it to differentiate $\sin(x)\cos(x)\tan(x)\log(x)\exp(x)$. It gave us the answer $(x\log(x)\sin(2x) - x\log(x)\cos(2x)/2 + x\log(x)/2 - \cos(2x)/2 + 1/2)\exp(x)/x$. As an AI researcher we want to find out whether this is correct. AI for example failed miserably to solve problem 7.4. Give at least two strategies to verify or falsify the output. You are allowed to use any tools, also AI ... What is the analysis of your expert opinion?

Solution:
Possible strategies:
1) compute it yourself and simplify to the expression. This is not so unreasonable as $\sin(x)\cos(x)\tan(x) = \sin^2(x)$ so that we have only to differentiate $\sin^2(x)\log(x)\exp(x)$ which can be done by hand.
2) Use Wolfram alpha to get an expression for the derivative, then check the graphs to see whether it is the same.
3) Compute the derivative using an other tool like Wolfram alpha or a calculator which can do symbolic calculations or with Mathematica.
4) Compute the derivative at some values and check whether the values agree.

P.S. Here is the Python code which Chat GPT suggested to answer problem 5):

```python
from sympy import sin, cos, tan, log, exp, symbols, diff
x=symbols('x')
f=sin(x)*cos(x)*tan(x)*log(x)*exp(x)
f_prime = diff(f,x)
print(f_prime.simplify())
```

P.P.S. Mathematica gave us for f_{10}, f_{20} (not yet differentiated) for problem 4:

$$\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{x+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}$$

$$\frac{1}{x+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}$$

And here is f_{50}, our Guinness book submission entry: (isn't it gorgeous?)

$$\frac{1}{x+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}$$

How do machines solve in this problem? We asked Mathematica to compute f_8'.

$$\frac{1}{(x+1)^2\left(\frac{1}{x+1}+1\right)^2\left(\frac{1}{\frac{1}{x+1}+1}+1\right)^2\left(\frac{1}{\frac{1}{\frac{1}{x+1}+1}+1}+1\right)^2\left(\frac{1}{\frac{1}{\frac{1}{\frac{1}{x+1}+1}+1}+1}+1\right)^2\left(\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{x+1}+1}+1}+1}+1}+1\right)^2\left(\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{x+1}+1}+1}+1}+1}+1}+1\right)^2\left(\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{x+1}+1}+1}+1}+1}+1}+1}+1\right)^2}$$

The program gave the output *Up yours*! when asked to give a LaTeX output for f_{50}'.

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