

INTRODUCTION TO CALCULUS

MATH 1A

Unit 9: Hospital

9.1. Hospital's rule is a fantastic tool. It allows to compute limits.¹ It is a miracle procedure and the answer to all our prayers to save us from dreadful limit computations!

Hospital's rule. If f, g are differentiable and $f(p) = g(p) = 0$ and $g'(p) \neq 0$, then

$$\lim_{x \rightarrow p} \frac{f(x)}{g(x)} = \lim_{x \rightarrow p} \frac{f'(x)}{g'(x)} .$$

Lets see how it works in examples:

The fundamental theorem of trigonometry:

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = \lim_{x \rightarrow 0} \frac{\cos(x)}{1} = 1 .$$

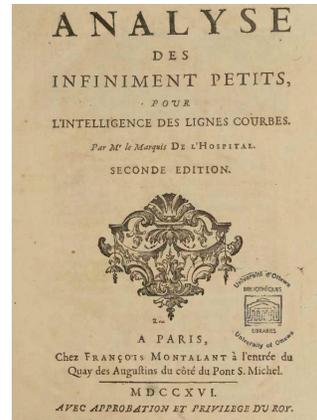
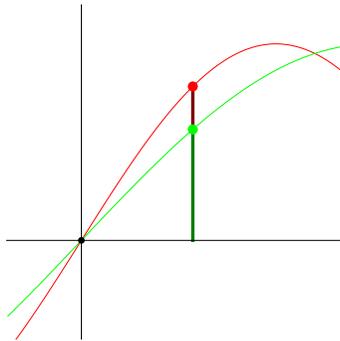
Note that this does not replace the derivation because it is equivalent to $\sin' = \cos$!

9.2. The proof of the rule is very simple: since $f(p) = g(p) = 0$ we have for the average rate of changes $Df(p) = (f(p+h) - f(p))/h = f(p+h)/h$ and $Dg(p) = (g(p+h) - g(p))/h = g(p+h)/h$ so that for every $h > 0$ with $g(p+h) \neq 0$. So, the **quantum l'Hospital rule** holds:

$$\frac{f(p+h)}{g(p+h)} = \frac{Df(p)}{Dg(p)} .$$

Now take the limit $h \rightarrow 0$. On the left we get $\lim_{h \rightarrow 0} f(x)/g(x)$ by definition. On the right we get $f'(p)/g'(p)$ by definition. Voilà!

¹Hospital is is easier to write and remember than Hôpital. Bring f to the hospital!



Problem. Find the limit $f(x) = (\exp(2x) - 1)/x$ for $x \rightarrow 0$.

Answer. The rule gives 2.

Problem. Find the limit $f(x) = \sin(100x)/\sin(101x)$ for $x \rightarrow 0$.

Answer. The rule 100/101.

9.3. The “first calculus book” was “Analyse des Infiniment Petits pour l’intelligence des Lignes Courbes” appeared in 1696. It was written by **Guillaume de l’Hospital** and has about 50-100 pages.² The mathematical content is mostly due to **Johannes Bernoulli**. The book remained the standard for a century.

9.4. Sometimes, we have to administer l’Hospital twice:

If $f(p) = g(p) = f'(p) = g'(p) = 0$ then $\lim_{x \rightarrow p} \frac{f(x)}{g(x)} = \lim_{x \rightarrow p} \frac{f''(x)}{g''(x)}$ if $g''(p) \neq 0$.

Problem: What do you get if you apply l’Hospital to the limit $[f(x + h) - f(x)]/h$ as $h \rightarrow 0$?

Answer: Differentiate both sides with respect to h! And then feel awesome!

What is the limit $\lim_{x \rightarrow 0} |x|^x$? This will provide the best answer to the question **What is 0^0 ?**

Find the limit $\lim_{x \rightarrow 2} \frac{x^2 - 4x + 4}{\sin^2(x - 2)}$.

Solution: this is a case where $f(2) = f'(2) = g(2) = g'(2) = 0$ but $g''(2) = 2$. The limit is $f''(2)/g''(2) = 2/2 = 1$.

²Stewart’s book with 1200 pages probably contains about 4 million characters, about 12 times more than l’Hospital’s book. The OCR text of l’Hospital’s book of 200 pages has 300’000 characters.

Homework

Problem 9.1: For the following functions, find the limits as $x \rightarrow 0$ using Hospital:

- a) $\sin(7x)/(5x)$
- b) $(\exp(16x) - 1)/(\exp(17x) - 1)$
- c) $\sin^2(8x)/\sin^2(5x)$
- d) $\frac{\tan(4x)}{3x}$
- e) $\sin(\sin(11x))/x$.

Solution:

All with l'Hospital:

- a) $7/5$
- b) $16/17$
- c) $(8/5)^2$
- d) $4/3$.
- e) 11 .

Problem 9.2: Luna, a new math chatbot, teaches itself limits but still makes mistakes and struggles with concepts. Please evaluate its answers to the right:

Problem	Luna's Reasoning
a) $\lim_{x \rightarrow \pi} \frac{\cos x}{x - \pi}$	$\lim_{x \rightarrow \pi} \frac{\cos x}{x - \pi}$ is by Hospital equal to $\lim_{x \rightarrow \pi} \frac{-\sin x}{1} = 0$
b) $\lim_{x \rightarrow 5} \frac{x-5}{\sqrt{x-5}} = 0$	Hospital: $\lim_{x \rightarrow 5} 1/(1/(2\sqrt{x-5})) = \lim_{x \rightarrow 5} 2\sqrt{x-5} = 0$
c) $\lim_{x \rightarrow 0} \frac{e^x - 1}{x}$	$\lim_{x \rightarrow 0} (e^x - 1) = 0$ and 0 over anything is 0, the limit is 0.
d) $\lim_{x \rightarrow 0} \ln x x = 0$	$\ln x \rightarrow -\infty$ and $x \rightarrow 0$. As it is not f/g , the limit DNE.
e) $\lim_{x \rightarrow 0} \ln 2x /\ln x $	Hospital gives $\lim_{x \rightarrow 0} (1/2x)/(1/x) = \lim_{x \rightarrow 0} 1/2 = 1/2$.

Solution:

- a) Hospital does not apply because it is not a $\frac{0}{0}$ limit.
- b) $0/0$ is an indefinite form and Hospital works. The argumentation is correct even so one could also just simplify.
- c) This would have to be brought to the Hospital. The argumentation without it does not work and is also wrong.
- d) If an expression does not match l'Hospital, it does not mean the limit does not exist. Indeed, we can write $(\ln|x|)/(1/x)$ and use l'Hospital: $f'/g' = 1/x/(-1/x^2) = x$ converges to 0. e) This is just a calculation error. The result is 1 not $1/2$.

Problem 9.3: Use l'Hospital to compute the following limits $x \rightarrow 0$:

- a) $\lim_{x \rightarrow 0} x / \ln |x|$
- b) $\ln |5x| / \ln |x|$.
- c) $4\text{sinc}'(x) = 4(\cos(x)x - \sin(x))/x^2$
- d) $\ln |1 + x| / \ln |2 + x|$.
- e) $(e^x - 1)/(e^{2x} - 1)$

Solution:

- a) 0. This is of the form $0/\infty$. There is no need for l'Hospital.
- b) 1. The derivatives on the top and bottom are both $1/x$.
- c) Take l'Hospital and cancel an x to get 0. d) No need for l'Hospital as the bottom is not 0. The result is 0.
- e) A nice case for l'Hospital. The result is $1/2$.

Problem 9.4: We have seen how to compute limits with healing. Fix the broken bones by bringing them to the Hospital at $x \rightarrow 1$:

- a) $\frac{x^{100}-1}{x^{22}-1}$.
- b) $\frac{\tan^2(x-1)}{(\cos(x-1)-1)}$

Solution:

- a) $100/22 = 50/11$.
- b) Use Hospital, then cancel out a $\sin(x - 1)$ to get -2 .

Problem 9.5: These problems need to be done during commercial breaks of the Super Bowl! If you fail to do so, you will be sent to the hospital by fierce 1a minions.

- a) Find the limit $\lim_{x \rightarrow 0} \frac{x}{\tan(6x)}$.
- b) Find the limit $\lim_{x \rightarrow 5} \frac{x^2-25}{x-5}$
- c) Find the limit $\lim_{x \rightarrow 0} \frac{1-e^x}{x-x^3}$.
- d) Find the limit $\lim_{x \rightarrow 0} \frac{\ln(1+9x)}{4x}$.
- e) Find the limit $\lim_{x \rightarrow 1} (x^7 - 1)/(x^3 - 1)$.

Solution:

- a) With Hospital $1/6$.
- b) Can also be done by healing. 10 .
- c) Nice case for Hospital: -1 .
- d) Nice case for Hospital: $9/4$.
- e) Also perfect case for Hospital $7/3$.