

# INTRODUCTION TO CALCULUS

MATH 1A

## Unit 10: Infinity

**10.1.** This lecture is about infinity. The main point is that Hospital's rule for the **indefinite form** "0/0" works also for the **indefinite form** " $\infty/\infty$ " as well as when  $p = \infty$ . Given a function  $f(x)$ , we can look how  $f(x)$  grows when  $x \rightarrow \infty$ . If there is a limit for  $x \rightarrow \infty$ , we have a **horizontal asymptote**. For example  $\lim_{x \rightarrow \infty} \arctan(x) = \pi/2$ . We can also reach infinity vertically. If  $\lim_{x \rightarrow p} f(x)$  does not exist, there might be a **vertical asymptote**. The function  $f(x) = \frac{x^2+1}{x^2-1}$  for example has a horizontal asymptote  $y = 1$  as l'Hospital gives  $\lim_{x \rightarrow \infty} f(x) = 1$ . and vertical asymptotes at  $x = 1$  and  $x = -1$ . If  $\lim_{x \rightarrow p} f(x) = \infty$  and  $\lim_{x \rightarrow p} g(x) = \infty$  we can ask what happens with the limit of  $\lim_{x \rightarrow p} f(x)/g(x)$ . Again, this can be done with Hospital.

**Hospital's rule.** If  $f, g$  are differentiable and  $\lim_{x \rightarrow p} f(x) = \lim_{x \rightarrow p} g(x) = \infty$  and  $\lim_{x \rightarrow p} g'(x) \neq 0$ , then

$$\lim_{x \rightarrow p} \frac{f(x)}{g(x)} = \lim_{x \rightarrow p} \frac{f'(x)}{g'(x)}.$$

Find  $\lim_{x \rightarrow \infty} (7x^2 + x + 1)/(3x^2 - 1)$ . **Solution.** We check to have an indefinite form  $\infty/\infty$ . Differentiate both nominator and denominator to get  $\lim_{x \rightarrow \infty} (14x + 1)/6x$ . Having again an indefinite form  $\infty/\infty$ , we send it again to the Hospital. The answer is  $14/6 = 7/3$ .

Find  $\lim_{x \rightarrow 0} \log |7x|/\log |3x|$ . This is an indefinite form  $\infty/\infty$ . We can use l'Hospital and see  $\lim_{x \rightarrow 0} (7/7x)/(3/3x)$  which can be simplified to 1 for  $x \rightarrow 0$ .

**10.2.** About the proof: the case when both sides converge to infinity can be reduced to the 0/0 case by writing  $A = f/g = (1/g(x))/(1/f(x))$ . Use l'Hospital and take the derivative on both sides simplifies to  $(g'/f')A^2$ . Solving for  $A$  gives  $A = f'(p)/g'(p)$ .

Problem: Lets look at the limit  $\lim_{x \rightarrow \pi/2} \tan(3x)/\tan(7x)$ . First check this is an indefinite form  $\infty/\infty$ . Now take the derivatives on both sides:  $\lim_{x \rightarrow \pi/2} (3/\cos^2(3x))/(7/\cos^2(7x)) = 3/7$ .

# Homework

**Problem 10.1:** Lets look at the functions  $f(x) = \ln(x)$  and  $g(x) = x$ . In order to see which function grows faster, we study the limit  $\lim_{x \rightarrow \infty} f(x)/g(x)$ .

- What is  $f(x)/g(x)$  for  $x = e^{10}$  and  $x = e^{1000}$ ?
- Compute the limit  $\lim_{x \rightarrow \infty} f(x)/g(x)$  for  $x \rightarrow \infty$  using l'Hospital.

**Solution:**

- $f(e^{10}) = 10$  and  $f(e^{100}) = 100$ . So  $f/g = 10/e^{10}$  in the first case and  $f/g = 100/e^{100}$  in the second case.
- The limit is 0. l'Hospital gives  $\lim_{x \rightarrow \infty} = (1/x)/1 \rightarrow 0$ .

**Problem 10.2:** Let us introduce the notation  $f(x) \ll g(x)$  if  $\lim_{x \rightarrow \infty} f(x)/g(x) = 0$ . The meaning is that  $f(x)$  **grows asymptotically slower** than  $g(x)$ . For example,  $\sqrt{x} \sin(x) \ll x$  because  $\sqrt{x} \sin(x)/x \rightarrow 0$  for  $x \rightarrow \infty$ .

- Rank the functions  $e^x, x^x, \ln(x), \sqrt{x}, x, x^2$  with that order notation. Which one grows slowest, which is next etc, until which is grows fastest?
- Produce one single graphics that shows the graphs of all these 6 functions, plotted on the interval  $[0, 3]$ .

**Solution:**

$$\ln(x) \ll \sqrt{x} \ll x \ll x^2 \ll e^x \ll x^x.$$

**Problem 10.3:** Find the following limits involving the indefinite form  $\infty/\infty$ :

- $\lim_{x \rightarrow 0} \cot(x)/\cot(3x)$ .
- $\lim_{x \rightarrow \infty} \frac{3x^2+1}{4x^2+100}$ .
- $\lim_{x \rightarrow 0} \sqrt{\log(3x)}/\sqrt{\log(2x)}$ .
- Find  $\lim_{x \rightarrow \infty} (x^2 + x - 1)/\sqrt{5x^4 + 1}$ .

Hint to to c) and d): First square the expression and find the limit, then take the root of the result.

**Solution:**

- a) l'Hospital gives the limit  $1/\sin^2(x)/(3/\sin^2(3x)) = \sin^2(3x)/(3\sin^2(x))$ . First break it apart to get  $\lim_{x \rightarrow 0} \sin(3x)/\sin(x) \lim_{x \rightarrow 0} \sin(3x)/\sin(x)/3 = 3 * 3/3 = 3$ .
- b) Bring it twice to the hospital to get  $6/8$ .
- c) Compute the limit of the square. The square is  $\log(3x)/\log(2x)$  which by l'Hospital has the limit 1. So also the square root is 1. d) The squared version has the limit  $(4/20) = 1/5$ . So, the limit is  $1/\sqrt{5}$ .

**Problem 10.4:** Use l'Hospital to compute the following routine limits  $x \rightarrow \infty$  (we use for a change  $\log = \ln$ , which is the common notation in all computer programming languages and all higher mathematics).

- a)  $\log|x|/x$   
 b)  $\log|5x|/\log|x|$   
 c)  $x^2/(1+x^2)$   
 d)  $\log|1+x|/\log|2+x|$   
 e)  $(e^x - 1)/(e^{2x} - 1)$

**Solution:**

In each case, differentiate. Note that  $\frac{d}{dx} \ln(cx) = 1/x$ .

- a) 0  
 b) 1  
 c) 1  
 d) 1 e) 0 This can also be done by factoring out and see it as  $1/(e^x + 1)$ .

**Problem 10.5:** We have  $nx = x + x + x + \dots + x$  and  $x^n = x * x * x * \dots * x$ . In computer science, the Knuth arrow notation write this as  $x \uparrow n$ . The number  $3^4$  for example is  $3 * 3 * 3 * 3 = 256$ . Knuth then writes  $x \uparrow\uparrow n = x \uparrow x \uparrow \dots \uparrow x$  meaning to exponentiate  $n$  times. For example,  $x \uparrow\uparrow 4 = x^{x^{x^x}}$ .

- a) We have already studied  $f(x) = x^x = x \uparrow\uparrow 2$ . Lets look at  $g(x) = x \uparrow\uparrow 3 = x^{(x^x)}$ . Compute the numbers  $g(1), g(2), g(3)$  and if you have the tools, compute  $g(4)$ .
- b) What is the limit  $\lim_{x \rightarrow \infty} f(x)/g(x) = x^x/x^{(x^x)}$ ? You can use l'Hospital, but you need an idea.

**Solution:**

a)  $1^{1^1} = 1$ ,

$2^{2^2} = 2^4 = 16$ ,

$3^{3^3} = 3^{27} = 7625597484987$ ,

$4^{4^4} = 134078079299425970995740249982058461274793658205923933777235614437217640300735 46976801$

b) Write  $x^x = y$ , then compute the limit  $y/x^y \leq y/2^y$  for  $x > 2$ . Since  $y/2^y \rightarrow 0$ , we know that the limit is zero.

