

INTRODUCTION TO CALCULUS

MATH 1A

Unit 10: Infinity

10.1. This lecture is about infinity. The main point is that Hospital's rule for the **indefinite form** "0/0" works also for the **indefinite form** " ∞/∞ " as well as when $p = \infty$. Given a function $f(x)$, we can look how $f(x)$ grows when $x \rightarrow \infty$. If there is a limit for $x \rightarrow \infty$, we have a **horizontal asymptote**. For example $\lim_{x \rightarrow \infty} \arctan(x) = \pi/2$. We can also reach infinity vertically. If $\lim_{x \rightarrow p} f(x)$ does not exist, there might be a **vertical asymptote**. The function $f(x) = \frac{x^2+1}{x^2-1}$ for example has a horizontal asymptote $y = 1$ as l'Hospital gives $\lim_{x \rightarrow \infty} f(x) = 1$. and vertical asymptotes at $x = 1$ and $x = -1$. If $\lim_{x \rightarrow p} f(x) = \infty$ and $\lim_{x \rightarrow p} g(x) = \infty$ we can ask what happens with the limit of $\lim_{x \rightarrow p} f(x)/g(x)$. Again, this can be done with Hospital.

Hospital's rule. If f, g are differentiable and $\lim_{x \rightarrow p} f(x) = \lim_{x \rightarrow p} g(x) = \infty$ and $\lim_{x \rightarrow p} g'(x) \neq 0$, then

$$\lim_{x \rightarrow p} \frac{f(x)}{g(x)} = \lim_{x \rightarrow p} \frac{f'(x)}{g'(x)}.$$

Find $\lim_{x \rightarrow \infty} (7x^2 + x + 1)/(3x^2 - 1)$. **Solution.** We check to have an indefinite form ∞/∞ . Differentiate both nominator and denominator to get $\lim_{x \rightarrow \infty} (14x + 1)/6x$. Having again an indefinite form ∞/∞ , we send it again to the Hospital. The answer is $14/6 = 7/3$.

Find $\lim_{x \rightarrow 0} \log |7x|/\log |3x|$. This is an indefinite form ∞/∞ . We can use l'Hospital and see $\lim_{x \rightarrow 0} (7/7x)/(3/3x)$ which can be simplified to 1 for $x \rightarrow 0$.

10.2. About the proof: the case when both sides converge to infinity can be reduced to the 0/0 case by writing $A = f/g = (1/g(x))/(1/f(x))$. Use l'Hospital and take the derivative on both sides simplifies to $(g'/f')A^2$. Solving for A gives $A = f'(p)/g'(p)$.

Problem: Lets look at the limit $\lim_{x \rightarrow \pi/2} \tan(3x)/\tan(7x)$. First check this is an indefinite form ∞/∞ . Now take the derivatives on both sides: $\lim_{x \rightarrow \pi/2} (3/\cos^2(3x))/(7/\cos^2(7x)) = 3/7$.

Homework

Problem 10.1: Lets look at the functions $f(x) = \ln(x)$ and $g(x) = x$. In order to see which function grows faster, we study the limit $\lim_{x \rightarrow \infty} f(x)/g(x)$.

- What is $f(x)/g(x)$ for $x = e^{10}$ and $x = e^{1000}$?
- Compute the limit $\lim_{x \rightarrow \infty} f(x)/g(x)$ for $x \rightarrow \infty$ using l'Hospital.

Problem 10.2: Let us introduce the notation $f(x) \ll g(x)$ if $\lim_{x \rightarrow \infty} f(x)/g(x) = 0$. The meaning is that $f(x)$ **grows asymptotically slower** than $g(x)$. For example, $\sqrt{x} \sin(x) \ll x$ because $\sqrt{x} \sin(x)/x \rightarrow 0$ for $x \rightarrow \infty$.

- Rank the functions $e^x, x^x, \ln(x), \sqrt{x}, x, x^2$ with that order notation. Which one grows slowest, which is next etc, until which is grows fastest?
- Produce one single graphics that shows the graphs of all these 6 functions, plotted on the interval $[0, 3]$.

Problem 10.3: Find the following limits involving the indefinite form ∞/∞ :

- $\lim_{x \rightarrow 0} \cot(x)/\cot(3x)$.
- $\lim_{x \rightarrow \infty} \frac{3x^2+1}{4x^2+100}$.
- $\lim_{x \rightarrow 0} \sqrt{\log(3x)}/\sqrt{\log(2x)}$.
- Find $\lim_{x \rightarrow \infty} (x^2 + x - 1)/\sqrt{5x^4 + 1}$.

Hint to to c) and d): First square the expression and find the limit, then take the root of the result.

Problem 10.4: Use l'Hospital to compute the following routine limits $x \rightarrow \infty$ (we use for a change $\log = \ln$, which is the common notation in all computer programming languages and all higher mathematics).

- $\log |x|/x$
- $\log |5x|/\log |x|$.
- $x^2/(1 + x^2)$.
- $\log |1 + x|/\log |2 + x|$.
- $(e^x - 1)/(e^{2x} - 1)$

Problem 10.5: We have $nx = x + x + x + \dots + x$ and $x^n = x * x * x * \dots * x$. In computer science, the Knuth arrow notation write this as $x \uparrow n$. The number 3^4 for example is $3 * 3 * 3 * 3 = 256$. Knuth then writes $x \uparrow \uparrow n = x \uparrow x \uparrow \dots \uparrow x$ meaning to exponentiate n times. For example, $x \uparrow \uparrow 4 = x^{x^{x^x}}$.

- We have already studied $f(x) = x^x = x \uparrow \uparrow 2$. Lets look at $g(x) = x \uparrow \uparrow 3 = x^{(x^x)}$. Compute the numbers $g(1), g(2), g(3)$ and if you have the tools, compute $g(4)$.
- What is the limit $\lim_{x \rightarrow \infty} f(x)/g(x) = x^x/x^{(x^x)}$? You can use l'Hospital, but you need an idea.

