INTRODUCTION TO CALCULUS

MATH 1A

Unit 11: Linearization

11.1. A differentiable function $f(x)$ can near a point a be approximated by

$$L(x) = f(a) + f'(a)(x-a) \ .$$

We call L the **linearization** of f near a. Why is L close to f near a? First of all, $L(a) = f(a)$. Next, we check that $L'(a) = f'(a)$. The functions L and f have not only the same function value, they also have the same slope at a.

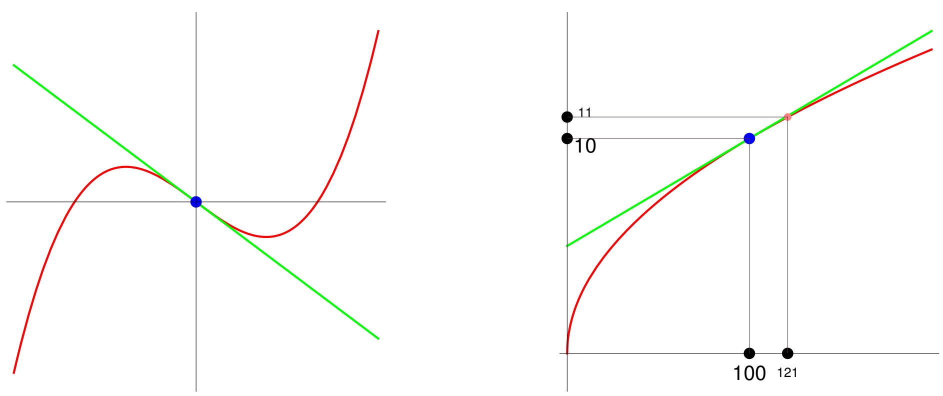

FIGURE 1. Left: The function $f(x) = x^3 - x$ and its linearization $y = f'(0)x + f(0) = -x$ at $a = 0$. Right: the function $f(x) = \sqrt{x}$ and its linearization $y = f'(100)x + f(100) = x/20 + 100$ at $a = 100$.

11.2. Lets look at $f(x) = x^2$ and $a = 10$. We have $f(10) = 100$ and $f'(x) = 2x$ which gives $f'(10) = 20$. The **linearization** = **linear approximation** of f at $a = 10$ is

$$L(x) = f(10) + f'(10)(x-10) = 100 + 20(x-10) = 20x - 100 \ .$$

Compare $f(x), L(x)$ near $a = 10$: we have $f(11) = 121$ and $L(11) = 20 \cdot 11 - 100 = 120$.

11.3. Why would we replace a fine function with something else? The reason is that the function $f(x)$ might be complicated. Lets take $f(x) = \sqrt{x}$ and try to compute $\sqrt{104}$ without a computer. We know that for $a = 100$, the square root can be computed as $f(a) = 10$. We also know that $f'(x) = 1/(2\sqrt{x})$ and so that $f'(a) = 1/(2 \cdot 10) = 1/20$. The linearization is now

$$L(x) = 10 + \frac{(x-100)}{20}, L(104) = 10 + \frac{4}{20} = 10 + \frac{1}{5} = \frac{51}{5} = 10.2 \ .$$

The actual number is 10.19804. Not too shabby, mate!

Homework due Friday Feb 16, 2024

Problem 11.1: Use linear approximation to estimate $\sqrt{103}$ and $\sqrt{97}$. Your results need to be fractions.

Solution:
We use $f(x) = \sqrt{x}$ which has the derivative $f'(x) = \frac{1}{2\sqrt{x}}$. We look at the nearest point, where the function can be computed easily. This is here $a = 100$, where $f'(a) = 1/20$.
a) $10 + 3/20$
b) $10 - 3/20$

Problem 11.2: a) Use linearization to find the 5th root of 34.
b) Impress your friends and compute the cube root of 1000001 to 10 digits in your head (of course using linearization!)

Solution:
a) $2 + (34 - 32)/(5 * 32^{4/5}) == 2 + 2/80$. b) $100 + 1/(3 \cdot 100^2) = 100 + 1/30000$.

Problem 11.3: You start a business by making **edible iphones**. You estimate that the cost for x cases will be $C(x)$ dollars, where the cost function is

$$C(x) = 20\sqrt{x} + 100$$

Its derivative $C'(x)$ is called the **marginal cost function**.
a) Use linearization to estimate $C(17)$.
b) Economists restate the notion of marginal cost by saying that $C'(x)$ is the cost of producing one more item when producing x items. Explain why this is not exactly true but why this is a reasonable statement.

Solution:
$C(16) = 180$. $C'(16) = 20/(2 * 4) = 5/2$. The solution is $180 + 2.5 = 182.5$.

Problem 11.4: Use linear approximation to estimate $e^{0.01}$. Is your estimate an overestimate or underestimate?

FIGURE 2. An edible iphone (AI generated).

Solution:
$f(x) = e^x$. We have $f(0) = 1, f'(0) = 1$. The estimate is $1 + 1(0.01 - 0) = 1.01$.

Problem 11.5: a) We all know that $\log'(x) = \ln'(x) = 1/x$. For $f(x) = \log_b(x)$ to the base b, we can use that $g(x) = e^{\log(b)x}$ has the derivative $g'(x) = \log(b)g(x)$. Argue using linearization to see that $\boxed{f'(x) = 1/(x\ln(b))}$.
b) Use linear approximation to estimate $\log_{10}(1000001)$.

Solution:
a) First note that by definition f and g are inverse of each other. Their graphs therefore are obtained from each other by flipping at $x = y$. What happens if we look at a linear function $y = mx + b$ has slope m, then $x = (y - b)/m$ has slope $1/m$. Since the linearizations have the same slope then the functions and the linearizations have reciprocal slopes, also the functions must have reciprocal slopes. So, remember that if you take the derivative of b^x, you just multiply the function with $\ln(b)$, if you take the derivative of $\log_b(x)$, you just divide $1/x$ by $\ln(b)$. b) We can now use part a) and use $a = 1000000$. We have the linear approximation $6 + 1/(a\ln(10)) = 6 + 1/(1000000\ln(10))$.

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