INTRODUCTION TO CALCULUS

MATH 1A

Unit 11: Linearization

11.1. A differentiable function $f(x)$ can near a point a be approximated by

$$L(x) = f(a) + f'(a)(x-a) \ .$$

We call L the **linearization** of f near a. Why is L close to f near a? First of all, $L(a) = f(a)$. Next, we check that $L'(a) = f'(a)$. The functions L and f have not only the same function value, they also have the same slope at a.

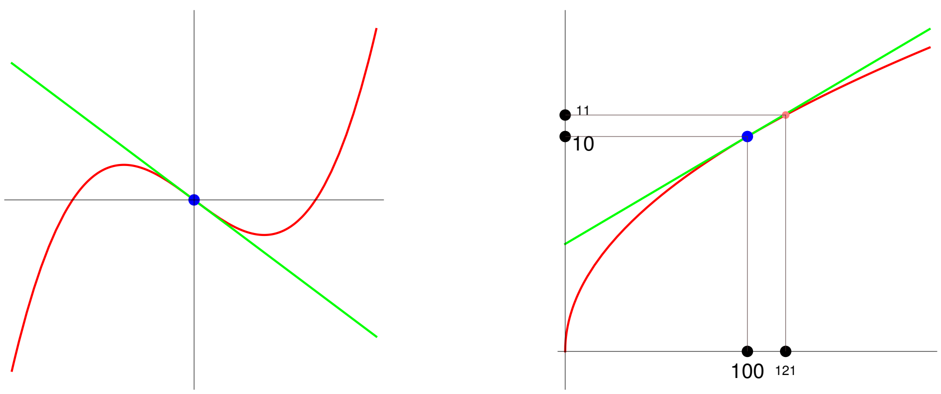

FIGURE 1. Left: The function $f(x) = x^3 - x$ and its linearization $y = f'(0)x + f(0) = -x$ at $a = 0$. Right: the function $f(x) = \sqrt{x}$ and its linearization $y = f'(100)x + f(100) = x/20 + 100$ at $a = 100$.

11.2. Lets look at $f(x) = x^2$ and $a = 10$. We have $f(10) = 100$ and $f'(x) = 2x$ which gives $f'(10) = 20$. The **linearization** = **linear approximation** of f at $a = 10$ is

$$L(x) = f(10) + f'(10)(x-10) = 100 + 20(x-10) = 20x - 100 \ .$$

Compare $f(x), L(x)$ near $a = 10$: we have $f(11) = 121$ and $L(11) = 20 \cdot 11 - 100 = 120$.

11.3. Why would we replace a fine function with something else? The reason is that the function $f(x)$ might be complicated. Lets take $f(x) = \sqrt{x}$ and try to compute $\sqrt{104}$ without a computer. We know that for $a = 100$, the square root can be computed as $f(a) = 10$. We also know that $f'(x) = 1/(2\sqrt{x})$ and so that $f'(a) = 1/(2 \cdot 10) = 1/20$. The linearization is now

$$L(x) = 10 + \frac{(x-100)}{20}, L(104) = 10 + \frac{4}{20} = 10 + \frac{1}{5} = \frac{51}{5} = 10.2 \ .$$

The actual number is 10.19804. Not too shabby, mate!

Homework due Friday Feb 16, 2024

Problem 11.1: Use linear approximation to estimate $\sqrt{103}$ and $\sqrt{97}$. Your results need to be fractions.

Problem 11.2: a) Use linearization to find the 5th root of 34.
b) Impress your friends and compute the cube root of 1000001 to 10 digits in your head (of course using linearization!)

Problem 11.3: You start a business by making **edible iphones**. You estimate that the cost for x cases will be $C(x)$ dollars, where the cost function is

$$C(x) = 20\sqrt{x} + 100$$

Its derivative $C'(x)$ is called the **marginal cost function**.
a) Use linearization to estimate $C(17)$.
b) Economists restate the notion of marginal cost by saying that $C'(x)$ is the cost of producing one more item when producing x items. Explain why this is not exactly true but why this is a reasonable statement.

Figure 2. An edible iphone (AI generated).

Problem 11.4: Use linear approximation to estimate $e^{0.01}$. Is your estimate an overestimate or underestimate?

Problem 11.5: a) We all know that $\log'(x) = \ln'(x) = 1/x$. For $f(x) = \log_b(x)$ to the base b, we can use that $g(x) = e^{\log(b)x}$ has the derivative $g'(x) = \log(b)g(x)$. Argue using linearization to see that $\boxed{f'(x) = 1/(x\ln(b))}$.
b) Use linear approximation to estimate $\log_{10}(1000001)$.

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