INTRODUCTION TO CALCULUS

MATH 1A

Unit 12: Maxima and Minima

12.1. Pierre Fermat made a simple but profound observation: if $f'(x)$ is not zero, then x can not be a maximum nor be a minimum. His reasoning was: if you make a step h then you end up at $f(x+h) \sim L(x+h) = f(x) + hf'(x)$. Indeed, we all know that if there is a slope and do a step we end up a bit higher.

12.2. Lets call a point x a **local maximum** of f if $f(y) \leq f(x)$ for all y near enough to x. The function $f(x) = x^3 - 2x$ for example has a local minimum at $x = 1$ and a local maximum at $x = -1$. The observation of Fermat is equivalent to:

Fermat's principle: If a differentiable function f has a local maximum or minimum at x, then $f'(x) = 0$.

12.3. The function $f(x) = x^2$ for example has the derivative $f'(x) = 2x$. This is zero at $x = 0$, the minimum of f. Note that the converse of Fermat's statement is not necessarily true: if $f'(x) = 0$, then x does not need to be a maximum or minimum. The standard example is $f(x) = x^3$. We have $f'(x) = 3x^2$ which is zero at $x = 0$. But $x = 0$ is neither a maximum nor minimum of f.

12.4. A point x is called a **critical point** of f, if $f'(x) = 0$. Critical points are important because they are **candidates for maxima and minima**.

12.5. The next test allows to see whether we have a maximum or minimum. The derivative should exists near a but not necessarily at a. Like for $f(x) = |x|$ and $a = 0$.

First derivative test: If a is a critical point of f and the slope $f'(x)$ changes from negative to positive at a then a is a local minimum. If $f'(x)$ changes from positive to negative at a, then a is a local maximum. If $f'(x)$ does not change sign, the a is neither a local maximum nor local minimum.

12.6. Second derivatives help. It assumes that the second derivative exists at a.

Second derivative test: If a is a critical point of f and $f''(a) > 0$, then f is a local minimum. If $f''(a) < 0$, then f is a local maximum. If $f''(a) = 0$, the test is inconclusive.

Homework

This PSet is due Wednesday February 21, 2024.

Problem 12.1: Find the critical points of the following two functions
a) $f(x) = x^4 - 4x^3 + 4x^2$.
b) $f(x) = x^4(x-3)^2$.

Solution:
a) The first derivative is $4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-2)(x-1)$ so that $x = 0, x = 1, x = 2$ are the critical points. b) The first derivative is $6x^5 - 30x^4 + 36x^3 = 6x^2(x^2 - 5x + 6)$ that we see the critical points $x = 0, x = 2, x = 3$.

Problem 12.2: Use the second derivative test to determine the nature of the critical points in the same two functions:
a) $f(x) = x^4 - 4x^3 + 4x^2$.
b) $f(x) = x^4(x-3)^2$.

Solution:
a) The second derivative of the function is $12x^2 - 24x + 8$. This is positive at $x = 0$, negative at $x = 1$ and positive at $x = 2$. The point $x = 0$ is a minimum, the point $x = 1$ is a maximum, the point $x = 2$ is again a minimum. b) The second derivative is $30x^4 - 120x^3 + 108x^2$. This is negative at $x = 2$ and positive at $x = 3$ and zero at $x = 0$. Therefore, we have a maximum at $x = 2$ a minimum at $x = 3$ and no conclusion at $x = 0$.

Problem 12.3: What does the first derivative test tell you about the behavior at the critical points in the two cases
a) $f(x) = x^4 - 4x^3 + 4x^2$.
b) $f(x) = x^4(x-3)^2$.
In each case, is there a point, where the first derivative test gives more information than the second derivative test?

Solution:
a) Look at the signs of $f'(x)$. In the first case we have a negative slope from $-\infty$ to 0, then a positive slope on $(0,1)$ and then a negative slope from $(1,2)$ and then again a positive slope. The conclusion of the first derivative test is the same.
b) In this case, the slope $f'(x)$ is positive until 0, but remains positive until 2, then becomes negative until 3 then again is positive. The point 0 is neither a maximum nor minimum. The conclusion at the points $x = 2$ and $x = 3$ are the same. In this case, the first derivative test tells more at $x = 0$. It assures that we have no maximum, nor any minimum at $x = 0$.

Problem 12.4: Find all the critical points and determine whether it is a local maximum, a local minimum or neither. You can use either test. You will see that some of the cases are a bit unusual.
a) $f(x) = e^2x - e^x$
b) $f(x) = e^x + x$
c) $f(t) = t^4 + t^3$
d) $f(t) = |2t - 8|$
e) $f(x) = 5$.

Solution:
a) Take the derivative $e^2 - e^x = 0$ gives $x = 2$. Use the second derivative test. Since the second derivative is positive, we have a minimum at $x = 2$.
b) There are no critical points because $1 + e^x$ is always positive.
c) Standard case. The derivative is $4t^3 + 3t^2$ which is zero at $t = 0$ and $t = -3/4$. Second derivative test is inconclusive at $x = 0$ as the second derivative is zero. We can use the first derivative test for $t = -3/4$. Since the second derivative is $12t^2 + 6t$ that is positive at $t = -3/4$, we have a minimum.
d) We need to use the first derivative test because the function is not differentiable at $t = 4$.
e) All points are critical points. All points are maxima and all points are also minima.

Problem 12.5: a) Both the first and second derivative test do not work for the **tamed devil function** $f(x) = x\sin(1/x)$ at $x = 0$. Why not? (Since we have no chain rule yet, you can certainly look up the first and second derivative using a tool like Wolfram alpha.).
b) Function $f(x) = \arcsin(\sin(x))$ has appeared in our ground hog movie. Where are the maxima and minima? To do so, plot the function $f(x)$ and its derivative $f'(x)$ and use one of the derivative tests at the maxima and minima.

Solution:
a) The derivative is $\sin(1/x) - \cos(1/x)/x$. This is not a continuous function. We can not even use the concept of critical point at a point of discontinuity.
b) If you plot it you see that the maxima are the same than the maxima of $\sin(x)$ and the minima are the same than the minima of $\sin(x)$. The maxima are at $\pi/2 + 2k\pi$. The minima are at $-\pi/2 + 2k\pi$.

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