Unit 14: Applications

14.1. Most laws in physics or chemistry are based on extremization: systems settle at minimal energy, systems maximize entropy, light minimizes action or soap bubbles minimize surface area. An other fundamental law of nature states that students optimize their day in order to have maximal time for practicing calculus.

14.2. A **soda can** is a cylinder of volume $\pi r^2 h$. Its **surface area** $2\pi rh + 2\pi r^2$ measures the amount of material used to manufacture the can. Assume the surface area is 2π, we can solve the equation for $h = (1 - r^2)/r = 1/r - r$ **Solution:** The volume is $f(r) = \pi(r - r^3)$. Find the can with maximal volume: $f'(r) = \pi - 3r^2\pi = 0$ showing $r = 1/\sqrt{3}$. This leads to $h = 2/\sqrt{3}$.

14.3. Take a piece of paper 2×2 inches. If we cut out 4 squares of equal side length x at the corners, we can fold up the paper to a tray with width $(2 - 2x)$ length $(2 - 2x)$ and height x. For which $x \in [0, 1]$ is the tray volume maximal?
Solution: The volume is $f(x) = (2 - 2x)(2 - 2x)x$. To find the maximum, we need to compare the critical points which is at $x = 1/3$ and the boundary points $x = 0$ and $x = 1$.

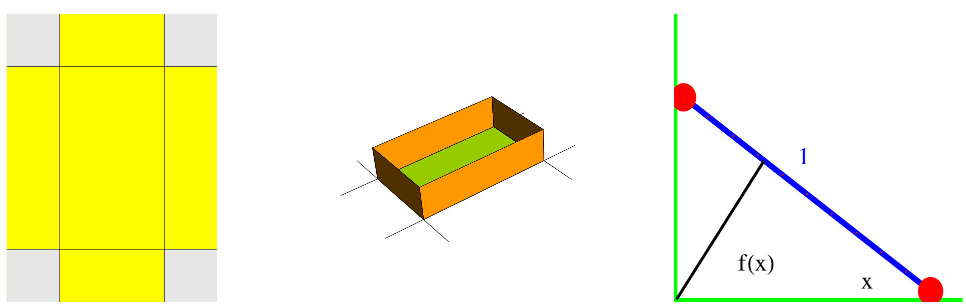


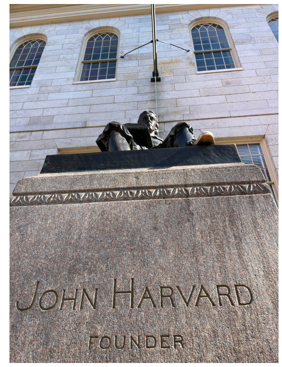
FIGURE 1. Finding the largest tray and the ladder problem.

14.4. Problem: A ladder of length 1 is one side at a wall and on one side at the floor. The distance of the ladder to the corner is $f(x) = \sin(x)\cos(x)$. [An other way to see this: the triangle has area $\sin(x)\cos(x)/2$. Since the hypotenuse is 1, the height must be $\sin(x)\cos(x)$.] Find the angle x for which $f(x)$ is maximal. **Solution** :

The distance is $f(x) = \sin(x)\cos(x) = \sin(2x)/2$ which has a local maximum when $f'(x) = 0$. The maximum is at $x = \pi/2$.

14.5. On February 21st, Oliver joined the tourists looking at the John Harvard Statue. For away or very close the viewing angle becomes small. There is an optimal distance, where the viewing angle is maximal. Find this distance. We work on this in a worksheet. Here are some pictures which show this. First from very far, then closer and closer and finally photographed from very close. Obviously there is an angle which is optimal.



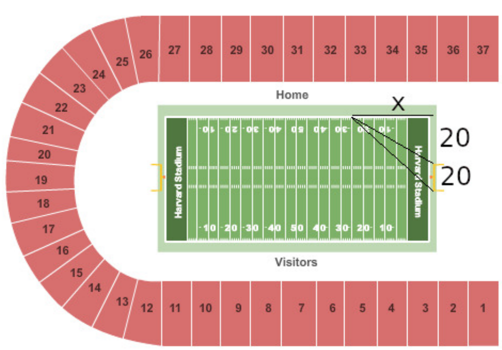


14.6. **Problem** We flip **coins**. The probability of hitting head is p. The entropy of this situation is defined as $S(p) = -p\log(p) - (1-p)\log(1-p)$. Which coin probability maximizes entropy? We will do this computation in class. Even more interesting is the minimization of **free energy** $F(p) = H - TS = ap + b(1-p) + Tp\log(p) + T(1-p)\log(1-p)$ which gives the **Gibbs distribution** $p = e^{b/T}/(e^{a/T} + e^{b/T})$.

Homework

This PSet is due Monday February 26, 2024.

Problem 14.1: Here is a problem very similar to the statue problem but a problem in football (as we have some football players in class): from which point on the sideline of the Harvard stadium does the goal post appear under the largest angle? We need to maximize $f(x) = \arctan(40/x) - \arctan(20/x)$.



Solution:
The derivative simplifies to $20(x^2 - 800)/[400 + x^2)(1600 + x^2)]$. The best solution is $x = \sqrt{800}$.

Problem 14.2: Which rectangle of dimensions x, y inscribed in $x^2/4 + y^2 = 1$ has maximal area $f = xy = 2x\sqrt{4 - x^2}$?

Solution:
The derivative is (again use technology if needed) $4(x^2 - 2)/\sqrt{4 - x^2}$. This is zero if $x = \sqrt{2}$. There is a natural condition that $x \geq 0$ and $x \leq 2$. But at thosse points the function is minimal zero.

Problem 14.3: **Mathcandy.com** manufactures spherical candies of effectiveness $f(r) = A(r) - V(r)$, where $A(r)$ is the surface area and $V(r)$ the volume of a candy of radius r. We want to have the largest effectivity of $f(r) = 4\pi r^2 - 4\pi r^3/3$.

Solution:
$f' = 8\pi r - 4\pi r^2$. This is zero for $r = \sqrt{2}$ and $r = 0$. The maximum is at $r = \sqrt{2}$.

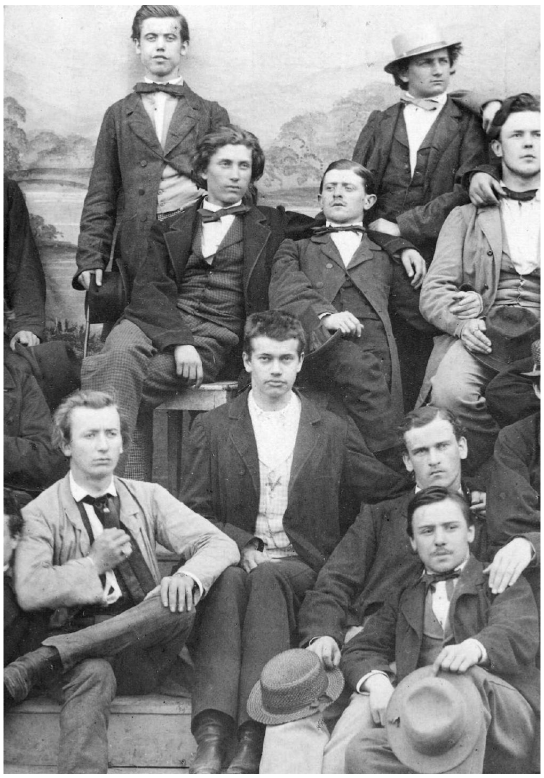
Problem 14.4: The function $S(x) = -x\ln(x)$ is called the **entropy** of the probability x. Find the probability $0 < x \leq 1$ which maximizes entropy.

Solution:
The derivative is $-1 - \ln(x)$. It is zero if $\ln(p) = -1$ which means $p = 1/e$. The maximal entropy is achieved for $p = 1/e$. The "Wahrscheinlichkeit" (Bolzmann was a German mathematician) is then $W = e$.

Problem 14.5: Find the global minimum of the **Helmholtz free energy** $G = H - TS$, where $T = 10$ is temperature, $S(x)$ is the entropy function in problem 14.4 and $H = x$ is an **internal energy**.
P.S. One of the most important principles in science is that nature tries to maximize entropy or minimize free energy.

Solution:
Take the derivative of derivative $G = x+10x\ln(x)$. The derivative is $1+10\ln(x)+10$. This is zero if $10\ln(x) = -11$ meaning $p = e^{-11/10}$. The minimal free energy is $\exp(-11/10)$.

Entropy has been introduced by Ludwig Boltzmann. It is important in physics and chemistry. $\boxed{S = k\log(W)}$ which one can find on his tombstone, is interpreted using "Wahrscheinlichkeit" $W(p) = 1/p$. Take the expectation of $\log(W)$ to get $S = -k\sum_p p\log(p)$. Note the use of log and not ln. Claude Shannon (a local) introduced the same entropy function in information theory. The picture to the right shows Hermann von Helmholz (1812-1894).



Boltzmann (1844-1906) Helmholtz (1812-1894).

Oliver Knill, knill@math.harvard.edu, Math 1a, Spring, 2024