INTRODUCTION TO CALCULUS

MATH 1A

Unit 15: Review

Overview

A function f is **continuous** at a if there is $b = f(a)$ such that $\lim_{x \to a} f(x) = b$. It is continuous on the interval $[a, b]$ if it is continuous on every point in $[a, b]$. The enemy of continuity are **jumps**, **infinity** and oscillation. The **first derivative** f' tells whether the function is **increasing** or **decreasing**. It is defined as the limit $[f(x+h) - f(x)]/h$ as $h \to 0$. The **second derivative** tells whether the function is **concave up**, **concave down**. Roots of f' are critical points. Roots of f'' can lead to inflection points, points where the concavity changes. The graph of the line $L(x) = f(a) + f'(a)(x-a)$ is tangent to the graph of f at a. A function is **even** if $f(-x) = f(x)$, and **odd** if $f(-x) = -f(x)$. If $f' > 0$ then f is **increasing**, if $f' < 0$ it is **decreasing**. If $f''(x) > 0$ it is **concave up**, if $f''(x) < 0$ it is **concave down**. If $f'(x) = 0$ then f has a **horizontal tangent**. To determine whether a point is a maximum or minimum, use either the **first derivative test** (change of f' near x) or the **second derivative test** (look at the sign of $f''(x)$). To maximize or minimze f on an interval $[a, b]$, find all critical points inside the interval, evaluate f on the **boundary** $f(a), f(b)$ and then compare the values to find the global maximum. If f is not differentiable somewhere, also include these **singular points** as candidates (like for $|x|$). To compute limits for indeterminate forms $0/0$ or ∞/∞, use **Hospital's theorem**: $\lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f'(a)/g'(a)$. To **estimate** $f(x)$ near a use **linearization** $f(x) \sim f(a) + f'(a)(x-a)$. A continuous function on $[a, b]$ has both a **global max** and global min by the **extreme value theorem**. The **fundamental theorem of trigonometry** is $\lim_{x \to 0} \sin(x)/x = 1$. To perform differentiation, master product and quotient rule.

Algebra reminders

Healing:	$(a+b)(a-b) = a^2 - b^2$ or $1 + a + a^2 + a^3 + a^4 = (a^5 - 1)/(a-1)$
Denominator:	$1/a + 1/b = (a+b)/(ab)$
Exponential:	$(e^a)^b = e^{ab}$, $e^a e^b = e^{a+b}$, $a^b = e^{b \ln(a)}$
Logarithm:	$\ln(ab) = \ln(a) + \ln(b)$. $\ln(a^b) = b \ln(a)$
Trig functions:	$\cos^2(x) + \sin^2(x) = 1, \sin(2x) = 2\sin(x)\cos(x), \cos(2x) = \cos^2(x) - \sin^2(x)$
Square roots:	$a^{1/2} = \sqrt{a}$, $a^{-1/2} = 1/\sqrt{a}$

Important functions

Polynomials	$x^3 + 2x^2 + 3x + 1$	Exponential	$5e^{3x}$
Rational functions	$(x+1)/(x^3 + 2x + 1)$	Logarithm	$\ln(3x)$
Trig functions	$2\cos(3x)$	Inverse trig functions	$\arctan(x)$

Important derivatives

$f(x)$	$f'(x)$
$f(x) = c$	0
$f(x) = x^n$	nx^{n-1}
$f(x) = e^{ax}$	ae^{ax}
$f(x) = \cos(ax)$	$-a\sin(ax)$
$f(x) = \arctan(x)$	$1/(1+x^2)$

$f(x)$	$f'(x)$
$f(x) = 1/x$	$-1/x^2$
$f(x) = \sin(ax)$	$a\cos(ax)$
$f(x) = \tan(x)$	$1/\cos^2(x)$
$f(x) = \ln(x)$	$1/x$
$f(x) = \sqrt{x}$	$1/(2\sqrt{x})$

Differentiation rules

Scaling rule	$(d/dx f(cx) = f'(cx)$.	Translation rule	$(d/dx f(x+a) = f'(x+a)$.
Addition rule	$(cf+g)' = cf' + g'$.	Quotient rule	$(f/g)' = (f'g - fg')/g^2$.
Product rule	$(fg)' = f'g + fg'$.	Easy rule	simplify before deriving

Limit examples

$\lim_{x\to 0} \sin(x)/x$	l'Hospital 0/0	$\lim_{x\to 1}(x^2-1)/(x-1)$	heal
$\lim_{x\to 0}(1-\cos(x))/x^2$	l'Hospital 0/0 twice	$\lim_{x\to\infty}\exp(x)/(1+\exp(x))$	l'Hospital
$\lim_{x\to 0}(1/x)/\ln(x)$	l'Hospital ∞/∞	$\lim_{x\to 0}(x+1)/(x+5)$	no work necessary

Is $1/\ln|x|$ continuous at $x = 0$? Answer: yes with $f(0) = 0$

Is $\ln(1/|x|)$ continuous at $x = 0$. Answer: no.

$\lim_{x\to 1}(x^{1/3}-1)/(x^{1/4}-1)$. Answer: 4/3.

$\lim_{x\to 0} \frac{(e^x-1)(\sin(5x)}{e^{3x}-1}\sin(7x)$. Answer: 35/3.

$\lim_{x\to 0} \frac{x^{10000}-1}{x^{2000}-1}$. Answer: $10000/20000 = 5$.

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