

# INTRODUCTION TO CALCULUS

MATH 1A

## Unit 16: Mind hacks

**16.1.** Before we move on to a new topic, it is good to reflect a bit on the work we have done so far and also learn from the first midterm. Doing mathematics is not only about "adding content", but a "gym for your mind" in general. Even more important than content or "algorithmic knowledge" is "meta knowledge", knowing how to think and knowing how to work."



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**16.2.** In computer science lingo, we mean with the later that we want to "tweak our operating system". The analog of "knowledge" are the files in your computer. The analog of "doing computations" is "running programs". But a thousand times more important than these two things is how your knowledge is organized, accessed, found and linked.

**16.3.** Lets look at Unix, a powerful, simple and efficient operating system. Unix knows the paradigms simplicity, generality and clarity. It uses small units of procedures which can be combined to make larger things happen. The command "find" for example allows to find everything fast, "grep" allows to filter things quickly. The simple building blocks can then like lego pieces put together without having to build new programs for each combination. More importantly, programs can write programs. The program TeX for example which is used for processing this very text, has been written by an

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<sup>1</sup>illustration AI generated

other language WEB. Donald Knuth wrote a WEB program which produced pascal code for TeX.

## A top 10 list

**16.4.** I regularly reflect improving "thinking". This usually goes under the name "work habits".

- Principle:**
- 1) Learn from mistakes
  - 2) Improve work conditions
  - 3) Turn tricks into methods
  - 4) Use background processes
  - 5) Keep important things in memory
  - 6) List concepts that are not understood
  - 7) Talk it out, even with AI
  - 8) Keep mind maps
  - 9) Focus on the essentials
  - 10) Learn to absorb setbacks

## How to solve

**16.5.** The mother of all problem solving books is Polya's "How to solve it" which was published in 1945. If you read and absorb this book, you immediately get measurably stronger in math. Still after more than 70 years, it is the best. Here are the now famous **Polya principles**:

### **Polya principles**

1. **Understand** the problem: unknowns, data, draw figure.
2. Devise a **plan**: similar or related problem?
3. **Carry out** the plan: check each step.
4. **Examine** the solution: can other problems be solved as such?

**16.6.** Here is a problem taken from the book of Polya. We will learn how to solve this next time. It is related to the bottle calibration problem we have seen earlier in the class.

**Problem Problem::** Water is flowing with a constant rate of one cubic meter per second into a conical vessel  $x^2 + y^2 = z^2, z \geq 0$ . At which rate is the water level rising if the water depth is  $z$  meters?

**16.7.** Lets try in class to work on this problem. It will lead us to related rates covered next time.

16.8.

**Tao's deformation principles**

- a. Consider special, extreme or degenerate cases.
- b. Solve a simplified version of the problem
- c. Formulate a conjecture
- d. Derive intermediate steps which would get it.
- e. Reformulate, especially try contraposition.
- f. Examine solutions of similar problems
- g. Generalize the problem

## Homework

This PSet is due Monday March 4, 2024:

**Problem 1:** One of the continuity questions was whether  $x \ln |x|$  is continuous everywhere. Most of the class answered that the limit  $x = 0$  is not defined.

a) Use l'Hospital to show that  $\lim_{x \rightarrow 0} x \ln |x| = 0$ . This is of the form  $0 \cdot \infty$ . You need to write it in as an indefinite form  $\infty/\infty$  first.

b) Simplify  $e^{x \ln |x|}$  so that it has not more than 3 symbols. We have looked at this expression before.

**Solution:**

Use l'Hospital for  $\ln(x)/(1/x)$  (which is now an indefinite form)  $\infty/\infty$ . We have now the limit  $(1/x)/(-1/x^2) = -x$  which goes to zero. So, the limit is zero.

**Problem 2:** The table where one has to identify some features about functions was solved poorly. Please solve that problem again. Reflect for each topic "even, odd, periodic or invertible" why you were stuck.

**Solution:**

Function	1-periodic	odd	even	invertible
$\sin(2\pi x)$	*	*		
$\cos(2\pi x)$	*		*	
$(x - 1)^2$				
$x^3$		*		*
$e^x$				

**Problem 3:** Look over your first midterm and identify **one general point** which needs improvement. What is the most important concept or or more general pattern you need to work on for next time?

**Solution:**

This is a personal thing.

**Problem 4:** Solve the Polya problem mentioned in this text: "Water is flowing with a constant rate of one cubic meter per second into a conical vessel  $x^2 + y^2 = z^2, z \geq 0$ . At which rate is the water level rising if the water depth is  $z$  meters?" This is preparation for next week, when we will look at related rates problems.

**Solution:**

$V = \pi z^3/3$ . Taking the derivative gives  $1 = V' = (\pi 3z^2/3)z'$  so that  $z' = 1/(\pi z^2)$ .

**Problem 5:** Choose one of the **mind performance hacks** distributed in class (a selection from the book) Pick the one which does most appeal to you and summarize in a short paragraph.

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**Solution:**

Also this is personal.

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<sup>2</sup>For more literature, see <https://people.math.harvard.edu/knill/creativity/books.html>