INTRODUCTION TO CALCULUS

MATH 1A

Unit 18: Implicit Differentiation

18.1. Assume we have a relation between x and y like

$$x^4 y + x y^4 = 2x$$

and we also know that $x = 1$ and $y = 1$. Can we use this to get the derivative y' without actually solving for y?

18.2. The answer is yes. Just differentiate and use the chain rule:

$$4x^3 y + x^4 y' + y^4 + 4xy^3 y' = 2$$

Now solve for y' to get $y' = [2 - 4x^3 y - y^4]/[x^4 + 4xy^3]$. At $x = 1, y = 1$ we see the answer $-4/5$. This is really cool because we would not have been able to solve the above equation for y and differentiate that expression.

18.3. Lets look at the example $x^2 + 3y^2 = 4$. Can you find the derivative y' at $x = 1$ knowing $y = 1$? Solution. We have $2x + 6yy' = 0$, so that $y' = -2x/6y = -2/6 = -1/3$. In this case we would have been able to solve for y and differentiate.

18.4. A cool application of the chain rule is to find the derivatives of inverses: **Example:** What is $\log'(x)$? Lets pretend we do not know this already but that we know the derivative of e^x as well as that log is the inverse of $e^x = \exp(x)$. [1] **Solution** Differentiate the identity $\exp(\log(x)) = x$. On the right hand side we have 1. On the left hand side the chain rule gives $\exp(\log(x)) \log'(x) = x \log'(x)$. Setting this equal gives $x \log'(x) = 1$. Therefore $\log'(x) = 1/x$.

$$\frac{d}{dx} \log(x) = 1/x.$$

Denote by $\arccos(x)$ the inverse of $\cos(x)$ on $[0, \pi]$ and with $\arcsin(x)$ the inverse of $\sin(x)$ on $[-\pi/2, \pi/2]$ and with $\arctan(x)$ the inverse of $\tan(x)$. The arctan is defined everywhere.

[1] $\log(x)$ stands also for $\ln(x)$ "logarithmus naturalis". Similarly as $\exp(x) = e^x$ it abbreviates. Almost all computer languages (Python, C, Perl, R, Matlab, Mathematica) use "log" not "ln". Paul Halmos called "ln" a childish notation which no mathematician ever used. I fought against ln like Don Quixote for 20 years and gave up. Just assume $\ln = \log$ like $e^x = \exp(x)$.

18.5. Example: Find the derivative of $\arcsin(x)$. **Solution**.
1. Step) Start with $\sin(\arcsin(x)) = x$.
2. Step) Differentiate both sides using the chain rule $\cos(\arcsin(x))\arcsin'(x) = 1$.
3. Step) Isolate $\arcsin'(x)$: $\arcsin'(x) = 1/\cos(\arcsin(x))$.
4. Step) Simplify: $1/\cos(\arcsin(x)) = 1/\sqrt{1-\sin^2(\arcsin(x))} = 1/\sqrt{1-x^2}$.

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}, \arccos'(x) = -\frac{1}{\sqrt{1-x^2}}, \arctan'(x) = \frac{1}{1+x^2} \, .$$

Homework: Due 3/8/2024

Problem 18.1: You know $y^3 + x^2y + xy^2 = 14$. Find y' at $x = 1$ knowing $y = 2$.

Solution:
Differentiate $3y^2y' + 2xy + x^2y' + y^2 + 2xyy' = 0$. Solve for $(-y^2 - 2xy)/(x^2 + 3y^2 + 2xy)$. Plug in $x = 1$ and $y = 2$ to get $-8/17$.

Problem 18.2: a) Find the derivative of $f(x) = 1/x$ by differentiating $xf(x) = 1$. b) Compute $\text{arccot}'(x)$.

Solution:
a) Differentiate the identity $xf(x) = 1$. This gives $f(x) + xf'(x) = 0$. Now solve for $f'(x)$ to get $f'(x) = -f(x)/x$. Plugging in $f(x) = 1/x$ gives $f'(x) = -1/x^2$.
b) Differentiate the identity $\cot(\text{arccot}(x)) = x$ to get

$$\frac{-1}{\cos^2(\text{arccot}(x))}\text{arccot}'(x) = 1 \, .$$

Solve for $\text{arccot}'(x)$ to get

$$\text{arccot}'(x) = -\cos^2(\text{arccot}(x))$$

Now use the identity $\cos^2(x) = 1/(1 + \cot^2(x))$ to see

$$\text{arccot}'(x) = \frac{-1}{1+\cot^2(\text{arcot}(x))} = \frac{-1}{1+x^2}$$

Problem 18.3: a) Find the derivative of $f(x) = \sqrt{x}$ by differentiating $f(x)^2 = x$.
b) Find the derivative of $f(x) = x^{m/n}$ by differentiating $f(x)^n = x^m$.

Solution:
a) Differentiating $f(x)^2 = x$ gives $2ff' = 1$ so that $f' = 1/(2f) = 1/(2\sqrt{x})$.
b) Differentiating $f(x)^n = x^m$ gives $nf(x)^{n-1}f'(x) = mx^{m-1}$ so that $f'(x) = mx^{m-1}/(nf(x)^{n-1}) = mx^{m-1}/(nx^{m(n-1)/n}) = (m/n)x^{m-1-m(n-1)/n} = (m/n)x^{m/n-1}$.

Problem 18.4: a) What is the derivative of $\arcsin(\arccos(x))$?
b) What is the derivative of $\arctan(\arctan(x))$?
c) Compute the derivative of $\arctan(\arctan(\arctan(x)))$.
P.S. When drawing out the graphs of these iterations we see a limiting function.

Solution:
a) $\frac{1}{\sqrt{(1-\arccos^2(x))}} \frac{-1}{\sqrt{(1-x^2)}}$.
b) $\frac{1}{1+\arctan^2(x)} \frac{1}{1+x^2}$.
c) $\frac{1}{1+(\arctan(\arctan(x)))^2} \frac{1}{1+x^2} \frac{1}{1+\arctan^2(x)} \frac{1}{1+x^2}$

Problem 18.5: a) Compute $\operatorname{arccosh}'(x)$. b) Compute $\operatorname{arcsinh}'(x)$.

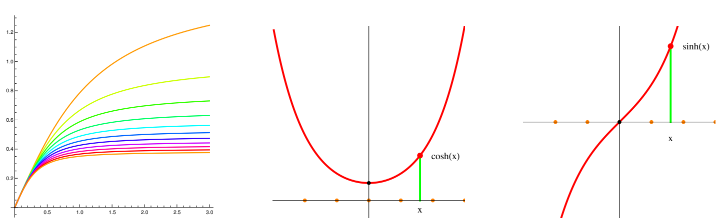

FIGURE 1. To the left, plots of $\arctan(x)$, $\arctan(\arctan(x))$, $\ldots \arctan(\arctan(\ldots \arctan(x)))$. In the middle, $\cosh(x) = (e^x + e^{-x})/2$, then $\sinh(x) = (e^x - e^{-x})/2$. We have $\cosh^2(x) - \sinh^2(x) = 1$.

Solution:
Use the identity $\cosh(x)^2 - \sinh^2(x) = 1$ we get $\cosh(x) = \sqrt{1 + \sinh^2(x)}$ and $\sinh(x) = \sqrt{\cosh^2(x) - 1}$.
a) Differentiate

$$\cosh(\text{arccosh}(x)) = x$$

to get $\text{arccosh}'(x) = 1/\sinh(\text{arccosh}(x))$. This is $1/\sqrt{\cosh^2(\text{arccosh}(x)) - 1} = 1/\sqrt{x^2 - 1}$.
b) Differentiate

$$\sinh(\text{arcsinh}(x)) = x$$

to get $\text{arcsinh}'(x) = 1/\cosh(\text{arcsinh}(x))$. This is $1/\sqrt{\cosh^2(\text{arccosh}(x)) + 1} = 1/\sqrt{x^2 + 1}$.