INTRODUCTION TO CALCULUS

MATH 1A

Unit 19: Related rates

19.1. Related rates problems solve relations between variables. Implicit differentiation is a special case. By taking derivatives of an equation, we get relations between variables. In all these problems, we have an $\boxed{\textbf{equation}}$ and a $\boxed{\textbf{rate}}$. You can then solve for the rate which is asked for.

19.2. Hydrophilic **water gel spheres** have volume $\boxed{V(r(t)) = 4\pi r(t)^3/3}$ and expand at a rate $\boxed{V' = 30}$. Find $r'(t)$. **Solution:** $30 = 4\pi r^2 r'$. We get $r' = 30/(4\pi r^2)$.

19.3. A **wine glass** has a shape $y = x^2$ and volume $\boxed{V(y) = y^2\pi/2}$. Assume we slurp the wine with constant rate $\boxed{V' = -0.1}$. With which speed does the height decrease? We have $d/dtV(y(t)) = V'(y)y'(t) = \pi y y'(t)$ so that $y'(t) = -1/(\pi y)$.

19.4. A **ladder** has length 1. Assume slips on the ground away with constant speed $\boxed{x' = 2}$. What is the speed of the top part of the ladder sliding down the wall at the time when $x = y$ if $\boxed{x^2(t) + y^2(t) = 1}$. Differentiation gives $2x(t)x'(t) + 2y(t)y'(t) = 0$. We get $y'(t) = -x'(t)x(t)/y(t) = 2 \cdot 1 = 1$.

19.5. A **kid slides** down a slide of the shape $\boxed{y = 2/x}$. Assume $\boxed{y'(t) = -7}$. What is $x'(t)$? Evaluate it at $x = 1$. **Solution:** differentiate the relation to get $y' = -2x'/x^2$. Now solve for x' to get $x' = -y'x^2/2 = 7/2$.

19.6. A **canister of oil** releases oil so that the area grows at a constant rate $\boxed{A' = 5}$. With what rate does the radius increase? **Solution.** See work sheet.

19.7. There is a saying: "everybody hates, related rates!". The reason is simple. If you look at related rates problems in textbooks, they are often hard to parse.

Related rates problems link quantities by a $\boxed{\text{rule}}$. These quantities can depend on time. To solve a related rates problem, differentiate the $\boxed{\text{rule}}$ with respect to time use the given $\boxed{\text{rate of change}}$ and solve for the unknown rate of change. To clarify, we have in this handout boxed the $\boxed{\text{rule}}$ and the known $\boxed{\text{rate of change}}$.

Homework: Due 3/18/2024

Problem 19.1: The **ideal gas law** $\boxed{pV = T}$ relates pressure p and volume V and temperature T. Assume the temperature $T = 50$ is fixed and $\boxed{V' = -3}$. Find the rate p' with which the pressure increases if $V = 10$.

Solution:
$-3 = V' = d/dt(T/p) = -(T/p^2)p' = -(50/p^2)p'$ so hat $p' = 3p^2/50$.

Problem 19.2: Assume the **total production rate** P of a new tablet computer product for kids is constant $P = 100$ and given by the **Cobb-Douglas formula** $\boxed{P = L^{1/3}K^{2/3}}$. Assume labor is increased at a rate $\boxed{L' = 2}$. What is the cost change K'? Evaluate this at $K = 125$ and $L = 64$.

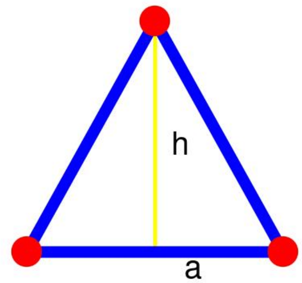
Solution:
$0 = P' = (1/3)L^{-2/3}K^{2/3}K' + (2/3)L^{1/3}K^{-1/3}2$. The answer is

$$K' = -\frac{(2/3)L^{1/3}K^{-1/3}}{(1/3)L^{-2/3}K^{2/3}} .$$

Problem 19.3: You observe an **airplane** at height $h = 10'000$ meters directly above you and see that it moves with rate $\boxed{\phi' = 5\pi/180 = \pi/36}$ radiants per second (which means 5 degrees per second). What is the speed x' of the airplane directly above you where $x = 0$? Hint: Use $\boxed{\tan(\phi) = x/h}$ to get ϕ for $x = 0$.

Solution:
Differentiate $(x/h) = \tan(\phi)$ to get $x'/h = \phi'/\cos^2(\phi) = 5\pi/180$. So that $x' = 5 * 10000\pi/180 = 872$ meters per second.

Problem 19.4: An **isosceles triangle** with base $2a$ and height h has fixed area $\boxed{A = ah = 1}$. Assume the height is decreased by a rate $\boxed{h' = -2}$. With what rate does a increase if $h = 1/2$?



Solution:
$A' = a'h + ah' = 0$ so that $a = -ah'/h = -2h'/(1/2) = -4h' = 8$.

Problem 19.5: There are **cosmological models** which see our universe as a four dimensional sphere which expands in space time. Assume the volume $\boxed{V = \pi^2 r^4/2}$ increases at a rate $\boxed{V' = 100\pi^2 r^2}$. What is r'? Evaluate it for $r = 47$ (billion light years).

Solution:
$V' = 2r^3\pi^2 r'$ so that $r' = 100\pi^2 r^2/(\pi^2 2r^3) = 50/(\pi^2 r) = 50/47$.

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