

INTRODUCTION TO CALCULUS

MATH 1A

Unit 19: Related rates

19.1. Related rates problems solve relations between variables. Implicit differentiation is a special case. By taking derivatives of an equation, we get relations between variables. In all these problems, we have an **equation** and a **rate**. You can then solve for the rate which is asked for.

19.2. Hydrophilic **water gel spheres** have volume $V(r(t)) = 4\pi r(t)^3/3$ and expand at a rate $V' = 30$. Find $r'(t)$. **Solution:** $30 = 4\pi r^2 r'$. We get $r' = 30/(4\pi r^2)$.

19.3. A **wine glass** has a shape $y = x^2$ and volume $V(y) = y^2\pi/2$. Assume we slurp the wine with constant rate $V' = -0.1$. With which speed does the height decrease? We have $d/dtV(y(t)) = V'(y)y'(t) = \pi y y'(t)$ so that $y'(t) = -1/(\pi y)$.

19.4. A **ladder** has length 1. Assume slips on the ground away with constant speed $x' = 2$. What is the speed of the top part of the ladder sliding down the wall at the time when $x = y$ if $x^2(t) + y^2(t) = 1$. Differentiation gives $2x(t)x'(t) + 2y(t)y'(t) = 0$. We get $y'(t) = -x'(t)x(t)/y(t) = 2 \cdot 1 = 1$.

19.5. A **kid slides** down a slide of the shape $y = 2/x$. Assume $y'(t) = -7$. What is $x'(t)$? Evaluate it at $x = 1$. **Solution:** differentiate the relation to get $y' = -2x'/x^2$. Now solve for x' to get $x' = -y'x^2/2 = 7/2$.

19.6. A **canister of oil** releases oil so that the area grows at a constant rate $A' = 5$. With what rate does the radius increase? **Solution.** See work sheet.

19.7. There is a saying: "everybody hates, related rates!". The reason is simple. If you look at related rates problems in textbooks, they are often hard to parse.

Related rates problems link quantities by a **rule**. These quantities can depend on time. To solve a related rates problem, differentiate the **rule** with respect to time use the given **rate of change** and solve for the unknown rate of change. To clarify, we have in this handout boxed the **rule** and the known **rate of change**.

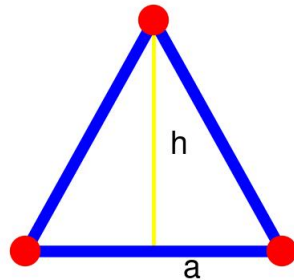
Homework: Due 3/18/2024

Problem 19.1: The **ideal gas law** $pV = T$ relates pressure p and volume V and temperature T . Assume the temperature $T = 50$ is fixed and $V' = -3$. Find the rate p' with which the pressure increases if $V = 10$.

Problem 19.2: Assume the **total production rate** P of a new tablet computer product for kids is constant $P = 100$ and given by the **Cobb-Douglas formula** $P = L^{1/3}K^{2/3}$. Assume labor is increased at a rate $L' = 2$. What is the cost change K' ? Evaluate this at $K = 125$ and $L = 64$.

Problem 19.3: You observe an **airplane** at height $h = 10'000$ meters directly above you and see that it moves with rate $\phi' = 5\pi/180 = \pi/36$ radians per second (which means 5 degrees per second). What is the speed x' of the airplane directly above you where $x = 0$? Hint: Use $\tan(\phi) = x/h$ to get ϕ for $x = 0$.

Problem 19.4: An **isosceles triangle** with base $2a$ and height h has fixed area $A = ah = 1$. Assume the height is decreased by a rate $h' = -2$. With what rate does a increase if $h = 1/2$?



Problem 19.5: There are **cosmological models** which see our universe as a four dimensional sphere which expands in space time. Assume the volume $V = \pi^2 r^4 / 2$ increases at a rate $V' = 100\pi^2 r^2$. What is r' ? Evaluate it for $r = 47$ (billion light years).