INTRODUCTION TO CALCULUS

MATH 1A

Unit 20: Intermediate value theorem

20.1. Finding solutions to an equation $g(x) = h(x)$ is equivalent to find roots of $f(x) = g(x) - h(x)$.

Definition: If $f(a) = 0$, then a is called a **root** of f. For $f(x) = \sin(x)$ for example, there are roots at $x = 0, x = \pi$.

20.2. The function $f(x) = \log|x| = \ln|x|$ has roots $x = 1$ and $x = -1$. The function $f(x) = e^x$ has no roots.

Intermediate value theorem of Bolzano. If f is continuous on the interval $[a, b]$ and $f(a), f(b)$ have different signs, then there is a root of f in (a, b).

20.3. The proof is constructive and important: we can assume $f(a) < 0$ and $f(b) > 0$. The other case is similar. Look at $c = (a+b)/2$. If $f(c) < 0$, then take $[c, b]$ as the new interval, otherwise, take $[a, c]$. We get a new root problem on a smaller interval. Repeat the procedure. After n steps, the search is narrowed to an interval $[u_n, v_n]$ of length $2^{-n}(b-a)$. Continuity assures that $f(u_n) - f(v_n) \to 0$ and that $f(u_n), f(v_n)$ have different signs. Both point sequences u_n, v_n converge to a root of f.

Verify that the function $f(x) = x^{17} - x^3 + x^5 + 5x^7 + \sin(x)$ has a root. **Solution.** The function goes to $+\infty$ for $x \to \infty$ and to $-\infty$ for $x \to -\infty$. We have for example $f(10000) > 0$ and $f(-1000000) < 0$. Use the theorem.

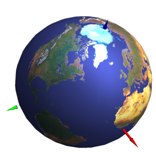
There is a solution to the equation $x^x = 10$. Solution: for $x = 1$ we have $x^x = 1 < 10$ for $x = 10$ we have $x^x = 10^{10} > 10$. Apply the intermediate value theorem to the function $f(x) = x^x - 10$.

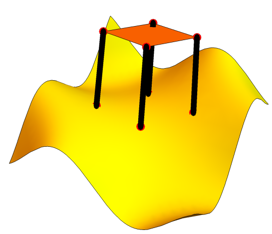
20.4. Earth Theorem. There is a point on the earth, where both temperature and pressure agree with the temperature and pressure on the antipode.

Proof. Draw an arbitrary meridian through the poles and let $f(x)$ be the temperature on that circle, where x is the polar angle. Define the function $g(x) = f(x) - f(x+\pi)$. If g is zero on the north pole, we have found our point. If not, $g(x)$ has different signs on the north and south pole. By the intermediate value theorem, there exists therefore an x, where $g(x) = 0$ and so $f(x) = f(x + \pi)$. For every meridian there is a latitude value $l(y)$ for which the temperature works. Define now $h(y) = l(y) - l(y + \pi)$. This function is continuous. Start with the meridian 0. If $h(0) = 0$ we have found our point. If not, then $h(0)$ and $h(\pi)$ have different signs. By the intermediate value theorem again, h has a root. There, both temperature and pressure agree with the antipode value.

20.5. Wobbly Table Theorem. On an arbitrary floor, a square table can be turned so that it does not wobble any more.

20.6. Proof. The 4 legs ABCD are located on a square in a plane. Let x be the angle of the line AC with with a coordinate axes if we look from above. Given x, the table can be positioned **uniquely**: the center of ABCD is on the z-axes, the legs ABC are on the floor and AC points in the direction x. Let $f(x)$ denote the height of the fourth leg D from the ground. If we find an angle x such that $f(x) = 0$, we have a position where all four legs are on the ground. Assume $f(0)$ is positive. ($f(0) < 0$ is similar.) Tilt the table around the line AC so that the two legs B,D have the same vertical distance h from the ground. Now translate the table down by h. This does not change the angle x nor the center of the table. The two previously hovering legs BD now touch the ground and the two others AC are below. Now rotate around BD so that the third leg C is on the ground. The rotations and lowering procedures have not changed the location of the center of the table nor the direction. This position is the same as if we had turned the table by $\pi/2$. Therefore $f(\pi/2) < 0$. The intermediate value theorem assures that f has a root between 0 and $\pi/2$.





20.7. The following is an application of the intermediate value theorem and also provides a constructive proof of the **Bolzano extremal value theorem** which we will see later.

Fermat's maximum theorem If f is continuous and has $f(a) = f(b) = f(a+h)$, then f has either a local maximum or local minimum inside the open interval (a, b).

20.8. The argument is to split the interval $[a, b]$ into two $[a, c]$ and $[c, b]$ of the same length. Now, $f(c) - f(a)$ and $f(b) - f(c)$ have different sign so that $g(x) = f(x + h/2) - f(x)$ has different signs $g(a)$ and $g(c)$ at the end points. By the intermediate value theorem there is a root of g in $[a, c]$ and therefore a point x in $[a, c]$ where $f(x) = f(x + h/2)$. This gives a new interval $[a_1, b_1]$ of half the size where the situation $f(a_1) = f(b_1)$ holds. Continuing like this we get a nested sequence of intervals $[a_n, b_n]$ which have size $2^{-n}h$. The limiting point is a maximum or minimum of f.

Homework: Due 3/20/2024

Problem 20.1: Find the roots for $-72 - 54x + 35x^2 + 15x^3 - 3x^4 - x^5$. You are told that all roots are integers.

Solution:
The roots are $-4, -3, -1, 2, 3$. They can be found by guessing the roots or plotting the function using desmos or mathematica and see them. Then check $(-4-x)(-2+x)(-3+x)(1+x)(3+x)$ is equal to the function.

Problem 20.2: Use the intermediate value theorem to verify that $f(x) = x^7 - 6x^6 + 8$ has at least two roots on $[-2, 2]$.

Solution:
At $x = 0$ the function is positive with value 8, at $x = -2$ the function is negative. At $x = 2$ the function is negative too. By the intermediate value theorem, there is a root between $x = 0$ and $x = -2$ as well as a root between $x = 0$ and $x = 2$.
By the way, since the function goes to infinity for $x \to \infty$, there would be even an other root between 2 and infinity but we have looked at the function only between -2 and 2.

Problem 20.3: The "Queen's gambit" features two fine actors Anya Taylor-Joy and Thomas Brodie-Sangster (both sharing expressive wide eyes). Anya's height is 170 cm, Thomas height is 178 cm. Anya was born April 16, 1996, Thomas was born on May 16, 1990. Anya's and Thomas net worth are both estimated to be 3 Million.
a) Can you argue that there was a moment when Anya's height is exactly half of Thomas height?
b) Can you argue that there was a moment when Anya's age was exactly half the age of Thomas?
c) Can you argue that there as a moment when Anya's fortune was exactly half of Thomas fortune?
Argue with the intermediate value theorem or note a scenario where the statement is false.

Solution:
a) The height of a person is a continuous function in time. At the birth time the height is small (lets take zero). Define the function $f(x) = t(x) - 2a(x)$, where $a(x)$ is the height of Anya and $t(x)$ is the height of Thomas. We have $f(1996) = t(1996) - 2a(1996) = 6 * L - 0 > 0$, where L is the height of Thomas given in cm. And we also have $f(2024) = t(2024) - 2a(2024) = 178 - 2 * 170 < 0$. By the intermediate value theorem, there must have been a time when $f(x)$ is zero.
b) A similar argument also works when we take age: $f(1996) = t(1996) - 2a(1996) = 6 - 2 * 0 = 6 > 0$. $f(2021) = t(2021) - 2a(2021) = 31 - 2 * 25 = -19 < 0$.
c) Fortune can change discontinuously because money has no continuum values and additions or subtractions from the bank are always by a nonzero amount. We can not use the intermediate value theorem.

Problem 20.4: Argue why there is a solution to
a) $5 - \sin(x) = x$, b) $\exp(7x) = x$, c) $\sin(x) = x^4$.
d) Why does the following argument not work:
The function $f(x) = 1/\cos(x)$ satisfies $f(0) = 1$ and $f(\pi) = -1$. There exists therefore a point x where $f(x) = 0$.
e) Does the function $f(x) = x + \log|\log|x||$ have a root somewhere? Argue with the intermediate value theorem.

Solution:
a) Look at $f(x) = 5 - \sin(x) - x$. It is positive for $x = 0$ and negative for $x = 100$. There exists a point where f is zero.
b) There is no root for $e^{7x} - x$.
c) The function $\sin(x) - x^4$ is positive 1 for $x = 0$ and negative for $x = 100$. d) Continuity fails at points $\pi/2$. We can not use the intermediate value theorem. e) Yes, for $x = e$ the value is e and so positive. For $x = a = e^{e^{-100}}$ the value is $x - 100 \sim -99$ which is negative. Note that x is very close to 1 but slightly larger. The function $f(x)$ is continuous for all $x > 1$.

Problem 20.5: a) Let $h = 1/2$. Find a h-critical point for the function $f(x) = |x|$. As defined in the text we look for a point for which $[f(x + h) - f(x)]/h = 0$.
b) Verify that for any $h > 0$, the function $f(x) = x^3$ has no h-critical point. There is no x, where $[f(x + h) - f(x)]/h = 0$ is possible.

Solution:
a) By definition there is a local h-maximum if $f(x + 1/2) = f(x)$. This is solved by $x = -1/4$. Indeed, if we are at $x = -1/4$ we get the same value as for $x = 1/4$.
b) The function $f(x) = x^3$ is monotone. We have $(x + h)^3 > f(x)$ for all x and so no critical point. [Here is the verification that x^3 is negative even so it is fine to assume it is obvious. It is easy to see after expansion that $(x + h)^3 > x^3$ for positive x For negative x it follows from symmetry as long as $x, x + h$ are both negative. If x is negative and $x + h$ is positive, then x^3 is negative and $(x + h)^3$ is positive so that also then we have monotonicity.]

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