INTRODUCTION TO CALCULUS

MATH 1A

Unit 22: Stability

22.1. In this lecture, we are interested how minima and maxima change when a parameter is changed. Nature, economies, or processes likes extrema. It turns out that if we change parameters, the outcome changes often in a non-smooth way. An economic parameter can change quickly for example. One calls this a catastrophe. This can be explained with mathematics. A key are **stable equilibria**, local minimum. Here is a general principle:

> If a local minimum disappears when we change an external parameter, the system settles in a new stable equilibrium. The new equilibrium can be far away from the original one.

22.2. To see this, let us look at the following optimization problem

22.3. Find all the minima and maxima of the function

$$f(x) = x^4 - x^2$$

Solution: $f'(x) = 4x^3 - 2x$ is zero for $x = 0, 1/\sqrt{2}, -1/\sqrt{2}$. The second derivative is $12x^2 - 2$. It is negative for $x = 0$ and positive at the other two points. We have two local minima and one local maximum.

22.4. Now find all the extrema of the function

$$f(x) = x^4 - x^2 - 2x$$

There is only one critical point. It is $x = 1$. Lets introduce $f_c(x) = x^4 - x^2 - cx$. The first function was $f_0(x)$, the second function was $f_2(x)$.

22.5. When the first graph is morphed into the second example, the local minimum to the left has disappeared. Assume the function f measures the prosperity of some kind and c is a **parameter**. We look at the position of the first critical point of the function. Catastrophe theorists look at the following **assumption**:

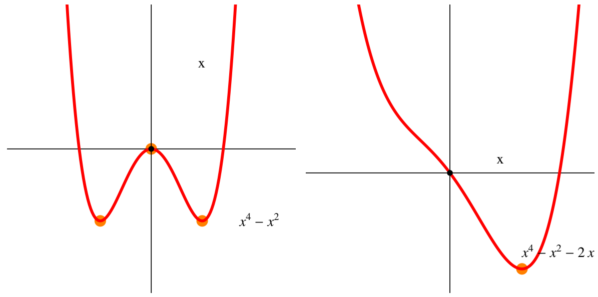
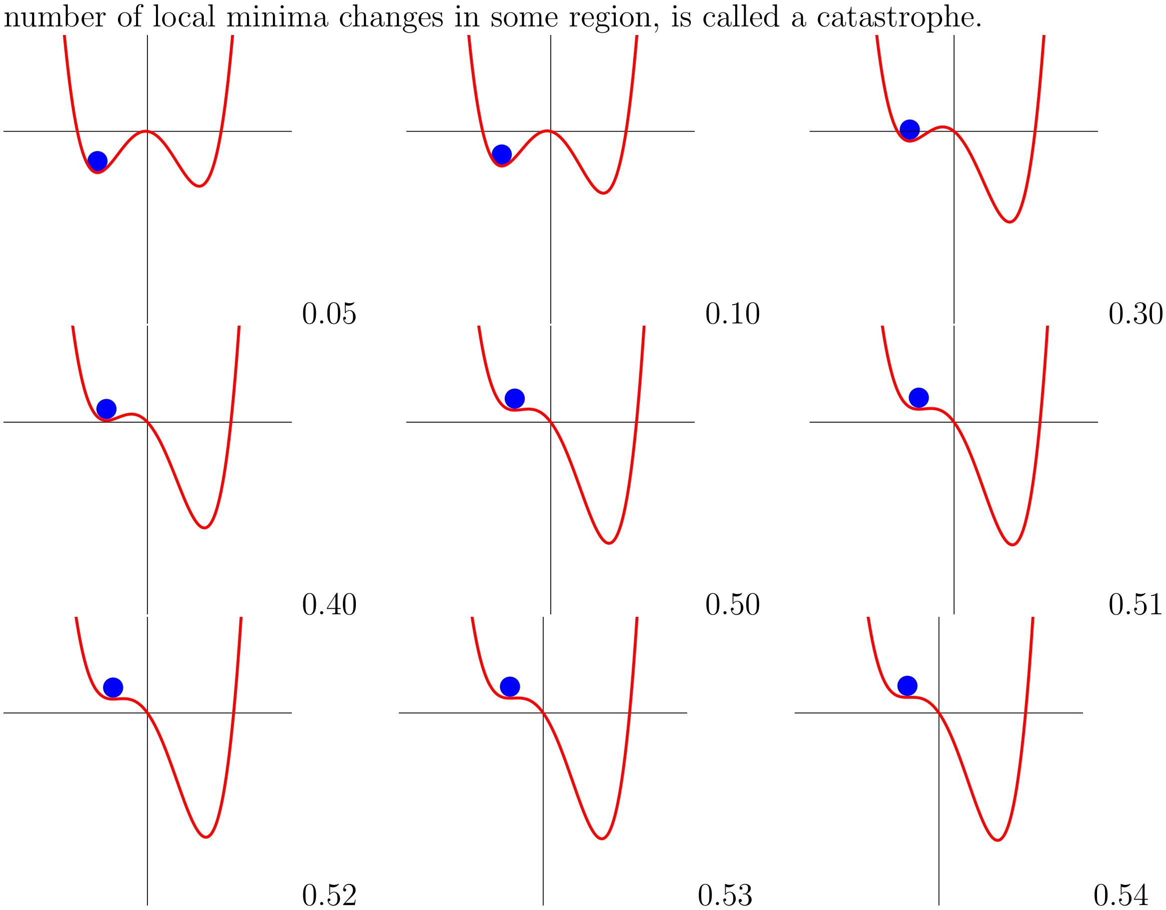


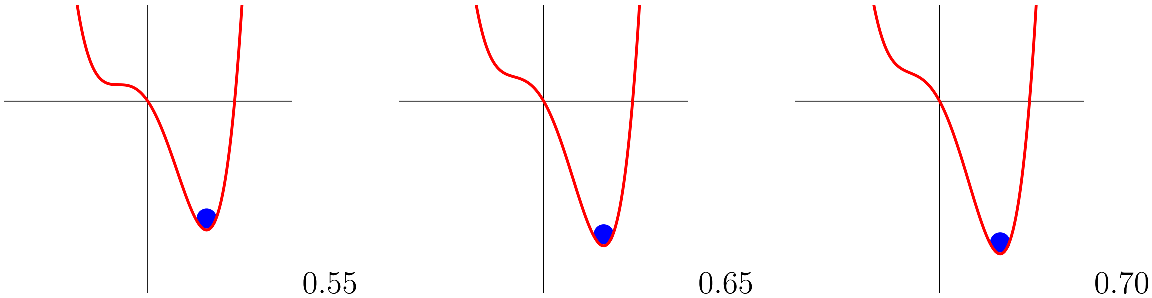
FIGURE 1. The function $f_0(x) = x^4 - x^2$ and $f_2(x) = x^4 - x^2 - 2x$.

22.6. Assume the function $f_c(x)$ depends on a parameter c the minimum, **stable equilibrium** depends on this parameter c. stable minimum disappears the system settles in general in an other stable equilibrium.

Definition: A parameter value c_0 at which somewhere a stable minimum disappears so that the system settles to an equilibrium away from it, is called a **catastrophe**.

22.7. In order to visualize a catastrophe, we draw the graphs of the function $f_c(x)$ for various parameters c and look at the local minima. At a parameter value, where the number of local minima changes in some region, is called a catastrophe.





22.8. A **bifurcation diagram** displays the equilibrium points as they change in dependence of the parameter c. The vertical axes is the parameter c, the horizontal axes is x. At the bottom for $c = 0$, there are three equilibrium points, two local minima and one local maximum. At the top for $c = 1$ we have only one local minimum. Here is an important principle:

Catastrophes often lead to a strict and abrupt decrease of the minimal critical value. It is not possible to reverse the process in general.

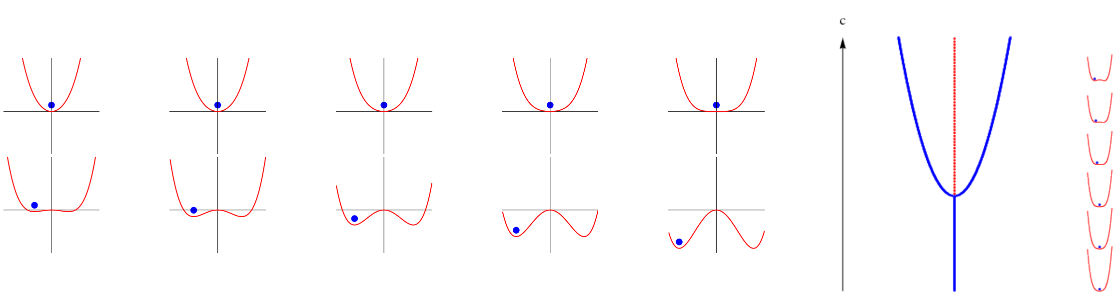
22.9. Let us look at this "movie" of graphs and run it backwards. By the same principle we do not end up at the position we started with. The new equilibrium remains the equilibrium nearby.

Catastrophes are in general **irreversible**.

22.10. We know this from experience: it is easy to screw up a relationship, reputation, get sick, have a ligament torn or lose somebody's trust. Building up a relationship, getting healthy or gaining trust usually happens continuously and slowly. Ruining the economy of a country or a company or losing a good reputation of a brand can be quick. It takes time to regain it.

Local minima can change discontinuously, when a parameter is changed. This can happen with perfectly smooth functions and smooth parameter changes.

We look at the example $f(x) = x^4 - cx^2$ with $-1 \leq c \leq 1$ in class.



Homework: Due Mar 25/2024

In this homework, we study a catastrophe for the function $f(x) = x^6 - x^4 + cx^2$, where c is a parameter between 0 and 1.

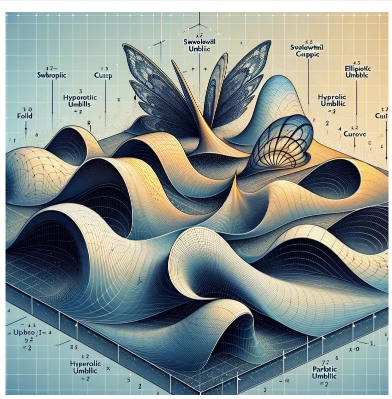
Problem 22.1: a) Find all the critical points in the case $c = 0$ and analyze their stability.
b) Find all the critical points in the case $c = 1$ and analyze their stability.

Problem 22.2: Plot the graph of the function $f(x)$ for $5 - 10$ values of c between 0 and 1. You can use desmos or Wolfram alpha. Mathematica example code is given in class.

Problem 22.3: If you change from $c = -1$ to 1, pinpoint the c value for which catastrophe (a discontinuous change of the minimum) occurs.

Problem 22.4: If you change back from $c = 0.6$ to -0.3, pinpoint the value for the catastrophe occurs. It will be different from the one in the previous question.

Problem 22.5: a) Write a 1000 word essay about "catastrophes" in math. In this problem you are officially allowed (and encouraged) to use AI. Be careful with prompting! Please also include what AI system you use and what your prompt was.
b) Repeat the exercise with a "one sentence definition". Again also include your prompt and the system that was used.
c) Have AI create an image about "catastrophe theory". Also here, be smart with prompting. You can use any tools. Include your picture and the tool you were using.



Here is a picture generated by chat GPT. It addresses more higher dimensional aspects of the theory in the context of singularity theory. That would be ok. Try to generate a picture which is closer to what we do here.

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