INTRODUCTION TO CALCULUS

MATH 1A

Unit 23: Riemann integrals

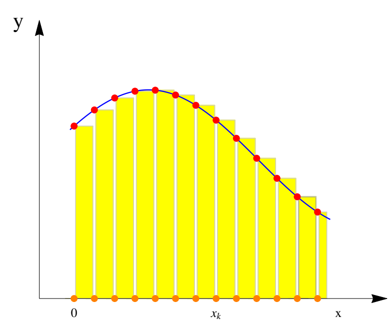
23.1. In this lecture, we define the **definite integral** $\int_a^b f(t)\,dt$ if f is a differentiable function. It has an interpretation as an **area under the curve**. Define $x_k = a + k\Delta x$ where $k = 0, \dots, n-1$ and $\Delta x = (b-a)/n$. The sum

$$S_n f = [f(x_0) + \cdots + f(x_{n-1})]\Delta x$$

is called a **Riemann sum**. It is a sum of areas of small rectangles of width Δx and height $f(x_k)$. It is a "left Riemann sum" because we evaluate the function to the left of the intervals.

Definition: Define

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\Delta x \ .$$



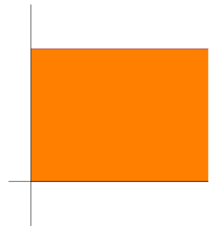
23.2. A very important result is that

For any differentiable function, the limit exists.

We can explicitly estimate the error: there are n little pieces where the region differs from the rectangle union. Each of these pieces has area $\leq M/n$, where M is the maximal slope that f can have in the given interval.

For non-negative f, the value $\int_0^x f(x)\,dx$ is the **area between the x-axis and the graph** of f. For general f, it is a **signed area**, the difference between two areas.

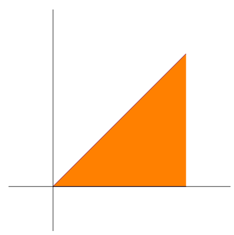
If $f(x) = c$ is constant, then $\int_0^x f(t)\,dt = cx$.



Let $f(x) = cx$. The area is half of a rectangle of width x and height cx so that the area is $cx^2/2$. Adding up the Riemann sum is more difficult. Let k be the largest integer smaller than $xn = x/h$. Then

$$S_n f(x) = \frac{1}{n}\sum_{j=1}^{k} \frac{cj}{n} = \frac{ck(k+1)/2}{n^2} \; .$$

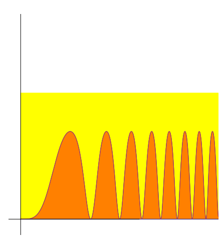
Taking the limit $n \to \infty$ and using that $k/n \to x$ shows that $\int_0^x f(t)\,dt = cx^2/2$.



Linearity of the integral $\int_a^b f(t) + g(t)\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt$ and $\int_a^b \lambda f(t)\,dt = \lambda \int_a^b f(t)\,dt$.

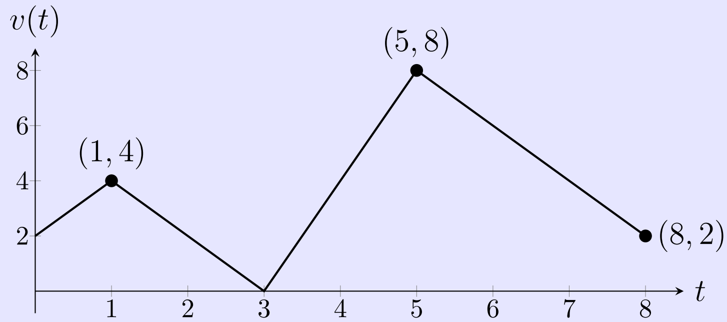
Upper bound: If $0 \leq f(x) \leq M$ for all x, then $\int_a^b f(t)\,dt \leq M(b-a)$.

$\int_0^x \sin^2(\sin(\sin(t)))/x\;dt \leq x$. **Solution**. The function $f(t)$ inside the interval is non-negative and smaller or equal to 1. The graph of f is therefore contained in a rectangle of width x and height 1.



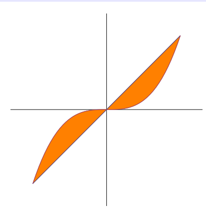
Homework due 3/27/2024

Problem 23.1: Below is the graph of the velocity of a bee traveling from a clover to a hive. Find the exact distance traveled by the bee between $t = 1$ and $t = 8$.



Problem 23.2: Lets look at the function $f(x) = \sin(x)$ on $[0, \pi]$.
a) Approximate the integral $\int_0^{\pi/2} \sin(x)\, dx$ using a Riemann sum with $\Delta x = \pi/4$.
b) Approximate the integral $\int_0^{\pi/2} \sin(x)\, dx$ using the Riemann sum with $\Delta x = \pi/6$.

Problem 23.3: The region enclosed by the graph of x and the graph of x^5 has a propeller type shape. Approximate its area by a Riemann sum using a Riemann sum with $\Delta x = 1/4$. It is your job to find a, b and n as well as the points $x_k = a + k(b-a)/n$.



Problem 23.4: Explain each rules with a picture:

- $\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$.
- $\int_a^b f(x)\, dx - \int_a^b g(x)\, dx = \int_a^b f(x) - g(x)\, dx$.
- $\int_a^b \lambda f(x)\, dx = \lambda \int_a^b f(x)\, dx$.

Problem 23.5: In this problem, it is crucial that you plot the function first. Split the integral up into parts. Find $\int_{-1}^4 f(x)\, dx$ for $f(x) = |x - |x - 2||$. As in 23.1 do not use a Riemann sum here. You can compute the value exactly.

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