

INTRODUCTION TO CALCULUS

MATH 1A

Unit 25: Anti derivatives

25.1. The following statement follows from the fundamental theorem: take $a = 0, b = x$ and differentiate $\int_0^x f'(t) dt = f(x) - f(0)$ with respect to x . We get $\frac{d}{dx} \int_0^x f'(t) dt = f'(x)$. Replacing f' with f tells us:

$$\frac{d}{dx} \int_0^x f(t) dt = f(x)$$

Here is a geometric picture: pick a small h and $y \in [x, x+h]$ so that $f(y)h$ is $\int_x^{x+h} f(t)dt$

$$\left[\int_0^{x+h} f(t) dt - \int_0^x f(t) dt \right] \frac{1}{h} = \int_x^{x+h} f(t) dt \frac{1}{h} = f(y)h \frac{1}{h} = f(y).$$

Taking the limit $h \rightarrow 0$ gives the statement of the fundamental theorem. ¹

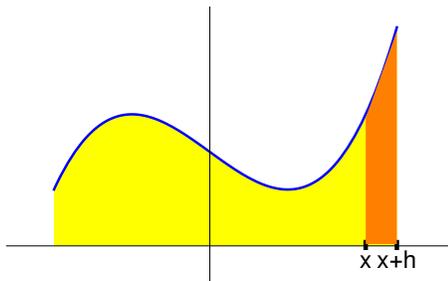


FIGURE 1. The fundamental theorem.

25.2. A function F with the property that $F'(x) = f(x)$ is called an **anti-derivative** of f . It is not unique but $F(x) = \int_0^x f(t) dt$ gives one anti-derivative. Define $\int_b^a f(x)dx = -\int_a^b f(x) dx$. This allows to define $F(x)$ for all x , also $x < 0$. Here is some more notation: we write an **indefinite integral** with a bare integral sign:

$$\int f(x) ; dx = F(x) + C ,$$

¹This argument could be modified to see that the Riemann integral and the fundamental theorem works even for continuous functions.

if $F(x)$ is an anti-derivative. The reason for adding the constant C is that the anti-derivative is not unique. We would write for example $\int \sin(x)dx = -\cos(x) + C$. Note that the constant will disappear if we compute a **definite integral**

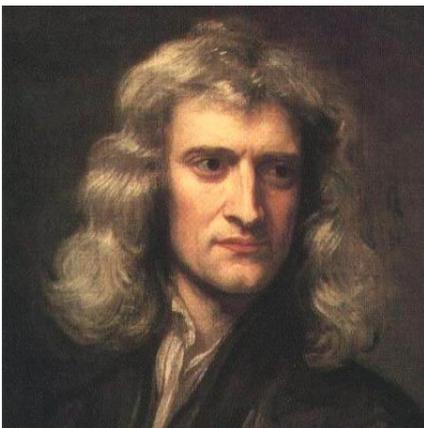
$$\int_a^b f(x) = F(b) - F(a)$$

if F is any anti-derivative. The **indefinite integral** is made up of some anti-derivative plus a constant.

25.3. You should have as many **anti-derivatives** “hard wired” in your brain. It really helps. Here are the core functions you should know.

| function | anti derivative |
|-------------------|-----------------------|
| 1 | x |
| x^n | $\frac{x^{n+1}}{n+1}$ |
| \sqrt{x} | $\frac{x^{3/2}}{3/2}$ |
| e^{cx} | $\frac{e^{cx}}{c}$ |
| $\cos(cx)$ | $\frac{\sin(cx)}{c}$ |
| $\sin(cx)$ | $-\frac{\cos(cx)}{c}$ |
| $\tan(cx)$ | $\ln(\cos(cx))/c$ |
| $\frac{1}{x}$ | $\ln(x)$ |
| $\frac{1}{1+x^2}$ | $\arctan(x)$ |
| $\ln(x)$ | $x \ln(x) - x$ |

25.4. Make your own table!



Newton (“Sir Slightly Annoyed”) and Leibniz (“Mr Sour Face”)

Isaac Newton and **Gottfried Leibniz** are the main figures for the fundamental theorem of calculus.

Homework: Due Apr 1/2024

Problem 25.1: Solve the following indefinite integrals:

a) $\int \sin(\sin(x)) \cos(x) dx$.

b) $\int \csc(x) \sec(x) dx$.

(rewrite $\csc(x) = 1/\sin(x)$, $\sec(x) = 1/\cos(x)$ so that the integrand is $\int \tan(x) \frac{1}{\cos^2(x)} dx$).

Solution:

a) $\sin(\sin(x)) + C$.

b) $\ln(\tan(x)) + C$.

Problem 25.2: Find anti derivatives of the following functions:

a) $1/(1+x)$

b) $1/(1+x^2)$

c) $x/(1+x^2)$

d) $x^2/(1+x^2)$ (will appear in class today)

e) $x^4/(1+x^2)$ (will appear in class today)

Solution:

a) $\ln(1+x) + C$.

b) $\arctan(x) + C$.

c) $\ln(1+x^2)^{\frac{1}{2}}$

d) Rewrite as $(x^2 + 1)/(1 + x^2) - 1/(1 + x^2)$. The anti-derivative is $x - \arctan(x) + C$.

e) Rewrite as $(x^4 - 1)/(1 + x^2) + 1/(1 + x^2) = (x^2 - 1) + 1/(1 + x^2)$. The anti-derivative is $x^3/3 - x + \arctan(x) + C$.

Problem 25.3: Evaluate the following indefinite integrals:

a) $\int 3^x + x^3 dx$

b) $\int 3^{3^x} 3^x dx$

Solution:

- a) Rewrite $e^x = e^{x \ln(3)}$. The answer is $e^{x \ln(3)} / \ln(3) + x^4/4$.
 b) From a) we have seen $\int 3^x dx = 3^x / \ln(3)$. So that the anti-derivative is $3^{3^x} / \ln(3)^2 + C$.

Problem 25.4: See whether you can write down the anti-derivatives in 5 minutes:

- a) $\ln(x)/x$
 b) $1/\sin^2(3x)$
 c) $\frac{1}{\cos(\ln(x))^2} \frac{1}{x}$
 d) $\tan(1 + 2x)$
 e) $\frac{\cos(x)}{2\sqrt{\sin(x)}}$

Solution:

- a) $\ln(x)^2/2 + C$
 b) $-\cot(3x)/3 + C$
 c) $\tan(\ln(x)) + C$
 d) $-\ln(\cos(1 + 2x))/2 + C$
 e) $\sqrt{(\sin(x))} + C$.

Problem 25.5: a) **A clever integral:** Evaluate the following integral (just by being clever, there is no algebra, and no work is needed):

$$\int_{-\pi}^{\pi} \sin(\sin(\sin(\sin(\sin(x)))))) dx .$$

b) **An evil integral:** Evaluate $\int_{e^e}^{e^{e^e}} \frac{1}{\ln(\ln(x)) \ln(x)x} dx$.

Solution:

- a) 0 because the function is odd. The area under the curve matches the area above the curve.
 b) The anti derivative is $\ln(\ln(\ln(x))) + C$. Evaluating gives $\ln(\ln(\ln(x)))|_{e^e}^{e^{e^e}} = 1 - 0 = 1$.