

# INTRODUCTION TO CALCULUS

MATH 1A

## Unit 28: Substitution

**28.1.** Functions like  $e^{6x}$  or  $1/(1+x)$  have been integrated by seeing  $\boxed{\int f(cx+a) dx = F(cx+a)/c}$ , if  $F' = f$ . We have also seen how to “spot the chain rule”: if  $f(x) = g(u(x))u'(x)$ ,  $\boxed{\int f(x) dx = G(u(x)) + C}$ , where  $G' = g$ .

**28.2. Example:**  $\int e^{x^4+x^2} (4x^3 + 2x) dx = e^{x^4+x^2} + C$ .

**Example:**  $\int \sqrt{x^5 + 1} x^4 dx = (2/15)(x^5 + 1)^{3/2}$ .

**Example:**  $\int \frac{\ln(x)}{x} dx = \ln(x)^2/2 + C$

**28.3.** The **method of substitution** formalizes this and makes it more systematic: **A)** select part of the formula, call it  $u$ . **B)** write  $du = u'dx$  and **C)** replace  $dx$  with  $du/u'$ . **D)** If all terms  $x$  have disappeared, integrate. **E)** Back substitute the variable  $x$ . If things should not work, go back to A) and try an other  $u$ .

$$\int f(u(x)) u'(x) dx = \int g(u) du .$$

**28.4. Example:** to get  $\int \ln(x)/x dx$ , pick  $u = \ln(x)$ , compute  $du = (1/x)dx$  and so  $dx = xdu$ . We get  $\int udu = u^2/2 + C$ . Back substitution gives  $\ln^2(x)/2 + C$ .

**Example:** to get  $\int \ln(\ln(x)) \frac{1}{\ln(x)} dx$ , try  $u = \ln(x)$  and  $du = (1/x)dx$ , then plug this in to get  $\int \ln(u)/u du = \ln^2(u)/2 + C$ . Back substitute to get  $\ln^2(\ln(x)) + C$ .

**Example:** To get  $\int \frac{x}{1+x^4} dx$ , substitute  $u = x^2$ ,  $du = 2xdx$  to get gives  $(1/2) \int du/(1+u^2)$   $du = (1/2) \arctan(u) = (1/2) \arctan(x^2) + C$ .

**Example:** To get  $\sin(\sqrt{x})/\sqrt{x}$ , try  $u = \sqrt{x}$ ,  $x = u^2$ ,  $dx = 2udu$ . The result is  $-2 \cos(\sqrt{x}) + C$ .

**28.5. Example:** (harder)  $\int \frac{x^3}{\sqrt{x^2+1}} dx$  might trigger the reflex  $u = \sqrt{x^2+1}$  but this does not work does not work. With  $u = x^2+1$  and  $du = 2xdx$  and  $dx = du/(2\sqrt{u-1})$ . We get

$$\int \frac{\sqrt{u-1}^3}{2\sqrt{u-1}\sqrt{u}} du = \int \frac{(u-1)}{2\sqrt{u}} = \int \frac{u^{1/2}}{2} - \frac{u^{-1/2}}{2} du = u^{3/2}/3 - u^{1/2} = \frac{(x^2+1)^{3/2}}{3} - (x^2+1)^{1/2} .$$

**28.6.** For **definite integrals**  $\int_a^b f(x) dx$ , one could find an anti-derivative as described and fill in the bounds.

$$\int_a^b g(u(x))u'(x) dx = \int_{u(a)}^{u(b)} g(u) du .$$

Proof. The right hand side is  $G(u(b)) - G(u(a))$  by the fundamental theorem of calculus. The integrand on the left has an anti derivative  $G(u(x))$ . Again by the fundamental theorem of calculus the integral leads to  $G(u(b)) - G(u(a))$ .

Example:  $\int_0^1 \frac{1}{5x+1} dx = [\ln(u)]/5|_1^6 = \ln(6)/5$ .

Example:  $\int_3^5 \exp(4x - 10) dx = [\exp(10) - \exp(2)]/4$ .

## Homework

**Problem 28.1:** Find the following anti-derivatives. But do it formally following the steps, even if you see the answer already.

a)  $\int x^2 \sin(x^3) dx$

c)  $\int -\cos(\sin(3x)) \cos(3x)/3 dx$

b)  $\int e^{x^6+x} (6x^5 + 1) dx$

d)  $\int e^{\tan(2x)} / \cos^2(2x) dx$ .

**Solution:**

a)  $-\cos(x^3)/3 + C$ .

b)  $e^{x^6+x} + C$ .

c)  $\sin(\sin(3x))$ .

d)  $e^{\tan(2x)}/2 + C$ .

**Problem 28.2:** Compute the following definite integrals. It is fine to find first an anti-derivative and only at the end place the bounds:

a)  $\int_1^2 \sqrt{x^5 + x} (5x^4 + 1) dx$

b)  $\int_0^{\sqrt{\pi}} 6 \sin(x^2) x dx$ .

c)  $\int_e^{e^2} \frac{\sqrt{\ln(x)}}{x} dx$

d)  $\int_0^1 \frac{5x}{\sqrt{1+x^2}} dx$ .

**Solution:**

- a) Substitute  $u = x^5 + x$ . The anti derivative is  $(2/3)(x + x^5)^{3/2}/5$ . Evaluating gives 130.283. b) The anti-derivative is  $-3 \cos(x^2)$ . The definite integral is 6.  
 c) The anti-derivative is  $2(\ln(x)^{3/2})/3$ . The definite integral is 1.2189.  
 d) The anti-derivative is  $5\sqrt{1+x^2}$ . The numerical answer is 2.071.

**Problem 28.3:** Compute  $\int_e^{2e} \frac{dx}{\sqrt{\ln(x)x}}$ .

**Solution:**

Substitute  $u = \ln(x)$ . The anti-derivative is  $2\sqrt{\ln(x)}$ . Numerically we get 0.60242.

**Problem 28.4:** Find the indefinite integrals:

a)  $\int \frac{x^5}{\sqrt{x^2+1}} dx$ .

b)  $\int \frac{1}{x(1+\ln(x)^2)} dx$

**Solution:**

- a) Substitute  $u = 1+x^2$ , then fill in  $x = \sqrt{u-1}$ . The expression is a bit more complicated. We end up with  $\sqrt{1+x^2}(3x^4 - 4x^2 + 8)/15$ .  
 b) Substitute  $u = \ln(x)$ . We get  $\arctan(\ln(x)) + C$ .

**Problem 28.5:** Find the anti-derivatives of a)  $\frac{\cos(x^3)}{e^{\sin(x^3)}} x^2$  b)  $\cot(\sqrt{x})/\sqrt{x}$ .

**Solution:**

Substitute  $u = \sin(x^3)$ . This gives  $\frac{1}{3}e^{-\sin(x^3)}$ . b)  $2\ln(\sin(\sqrt{x}))$ .