INTRODUCTION TO CALCULUS

MATH 1A

Unit 31: Extrema review: Economics

31.1. This week we review a few themes taught in this course. First we recall how to find maxima and minima. In economics, you want to minimize or maximize a quantity. Calculus expresses this using critical points. Two theorems decide whether we have minima or maxima. In economics, parameters are often external, like supply and demand, the interest rates etc. Given these parameters, one wants to extremize a quantity, like profit of a company. We have also looked at what happens if parameters are deformed. Deformation parameters are called "weights" in "transformer networks".

31.2. Calculus plays a pivotal role in **economics**. Lets also grasp the opportunity to get acquainted with some jargon in economics. Economists talk differently: $f' > 0$ means **growth** or **boom**, $f' < 0$ means **decline** or **recession**, a vertical asymptote is a **crash**, a horizontal asymptote is a **stagnation**, a discontinuity is labled as **"inelastic behavior"**, the derivative of something is the **"marginal"** of it. An example is marginal revenue.

Definition: The **marginal cost** is the derivative of the **total cost**.

31.3. The marginal cost and total cost are functions of the quantity x of goods:
Example: Assume the total cost function is $C(x) = 10x - 0.01x^2$. Use the marginal cost in order to minimize the total cost. **Solution**. Differentiate $C' = 10 + 0.02x$ At a minimum, this derivative is zero. Here at $x = 50$.

Example: You sell spring water. The marginal cost to produce it at time x (years) is $f(x) = 1000 - 2000\sin(x/6)$. For which x is the total cost maximal? **Solution**. We look for points where $F'(x) = f(x) = 0, F''(x) < 0$. This is the case for $x = \pi$. The cost is maximal in about 3 years.
In the book "Don't worry about Micro, 2008", Dominik Heckner and Tobias Kretschmer tell the following strawberry story (quoted in verbatim):

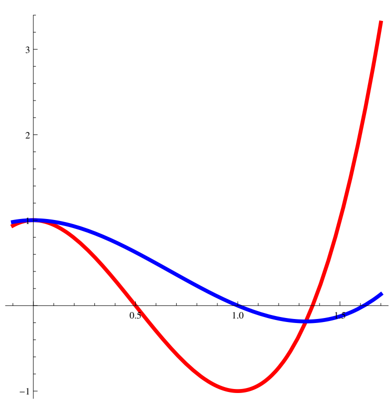
Suppose you have all sizes of strawberries, from very large to very small. Each size of strawberry exists twice except for the smallest, of which you only have one. Let us also say that you line these strawberries up from very large to very small, then to very large again. You take one strawberry after another and place them on a scale that sells you the average weight of all strawberries. The first strawberry that you place in the bucket is very large, while every subsequent one will be smaller until you reach the smallest one. Because of the literal weight of the heavier ones, average weight is larger than marginal weight. Average weight still decreases, although less steeply than marginal weight. Once you reach the smallest strawberry, every subsequent strawberry will be larger which means that the rate of decease of the average weight becomes smaller and smaller until eventually, it stands still. At this point the marginal weight is just equal to the average weight.

31.4. If $F(x)$ is the **total cost function** in dependence of the quantity x, then $F' = f$ is called the **marginal cost**.

Definition: The function $g(x) = F(x)/x$ is called the **average cost**.

Definition: A point, where $f = g$ is called a **break-even point**.

If $f(x) = 4x^3 - 3x^2 + 1$, then $F(x) = x^4 - x^3 + x$ and $g(x) = x^3 - x^2 + 1$. Find the break even point and the points, where the average costs are extremal. **Solution:** To get the break even point, we solve $f - g = 0$. We get $f - g = x^2(3x - 4)$ and see that $x = 0$ and $x = 4/3$ are two break even points. The critical point of g are points where $g'(x) = 3x^2 - 4x$. They agree:



31.5. The following theorem tells that the marginal cost is equal to the average cost if and only if the average cost has a critical point. Since total costs are typically concave up, we usually have "break even points are minima for the average cost". Since the strawberry story illustrates it well, lets call it the "strawberry theorem":

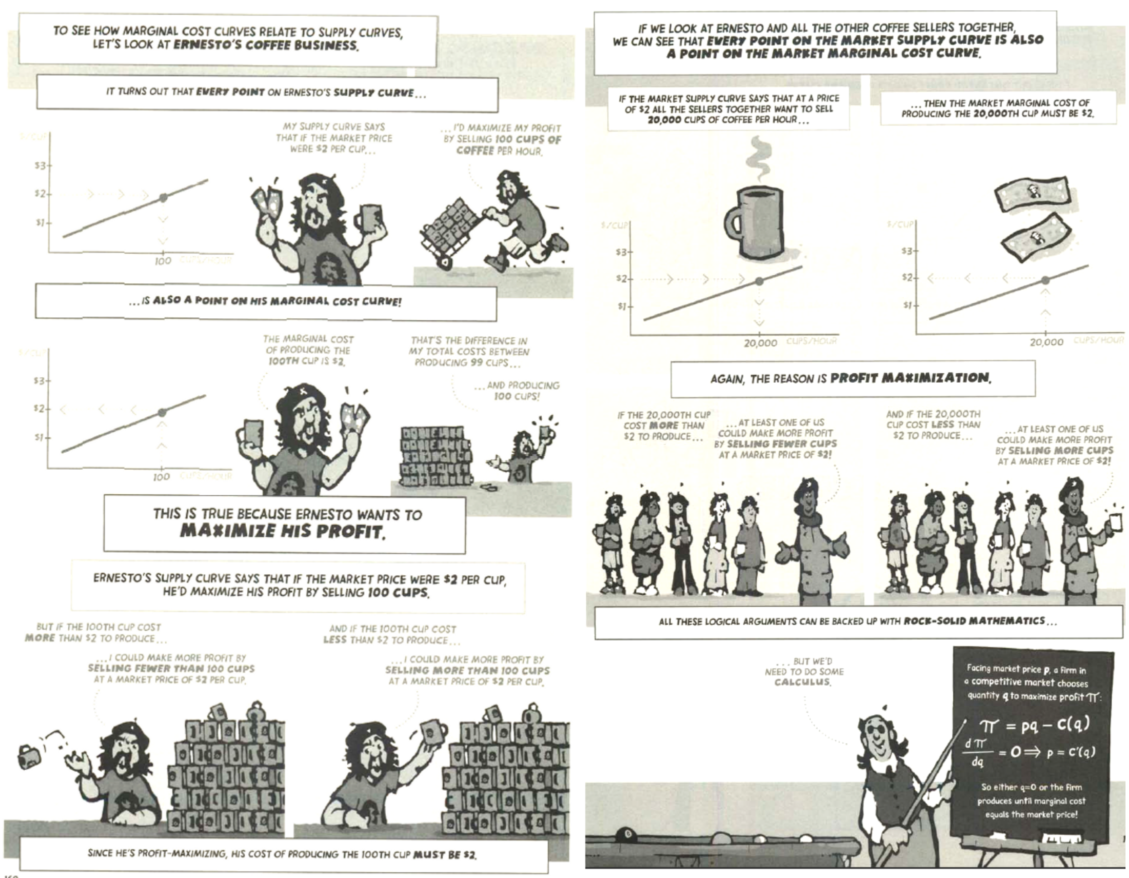
Strawberry theorem: $g'(x) = 0$ if an only if $f = g$.

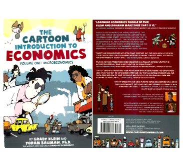
Proof. $g' = (F(x)/x)' = F'/x - F/x^2 = (1/x)(F' - F/x) = (1/x)(f - g)$.

More extremization

31.6. Lets review also some extremization problems:
[**Example:** Find the **rhomboid** with side length 1 which has maximal area. Use angle α to extremize.

Find the ellipse of length $2a$ and width $2b$ which has fixed area $\pi ab = \pi$ and for which the sum of diameters $2a + 2b$ is maximal. **Solution.** Find $b = 1/a$ from the first equation and plug into the second equation.





Source: Grady Klein and Yoram Bauman, The Cartoon Introduction to Economics: Volume One Microeconomics, published by Hill and Wang. You can detect the strawberry theorem ($g' = 0$ is equivalent to $f = g$) can be seen on the blackboard.

Homework

Problem 31.1: Find the break-even point for an economic system if the marginal cost is $f(x) = 1/x$.

Solution:
a) $F(x) = \log(x)$ and $g(x) = \log(x)/x$. The point $1/x == \log(x)/x$. This means $\boxed{x = e}$.
b) $F(x) = x^8/8$ and $g(x) = x^7/8$. The break even point is $x^7 = x^7/8$.

Problem 31.2: Let $f(x) = \cos(x)$. Compute $F(x)$ and $g(x)$ and verify that $f = g$ agrees with $g' = 0$.

Solution:
Compute $F(x) = \sin(x)$ and $g(x) = F(x)/x = \sin(x)/x$. The equation $f(x) = g(x)$ is $\cos(x) = \sin(x)/x$. We also compute $g'(x) = (xF' - F)/x^2 = -\cos(x)/x + \sin(x)$ and see that $g'(x) = 0$ if and only if $f(x) = g(x)$ (break even points). The point $x = 0$ is actually defined also for $g(x)$ because this is the sinc-function.

Problem 31.3: Time to review standard extremization: for smaller groups, production usually increases when adding more workforce. After some time, bottle necks occur, not all resources can be used at the same management and bureaucracy is added. We make a model to find the maximal production parameters. The **production function** in an office gives the production $Q(L)$ in dependence of labor L. Assume

$$Q(L) = 5000L^3 - 3L^5 .$$

Find L which gives the maximal the production. Use the second derivative test.

Solution:
The derivative is $Q'(L) = 150000L^2 - 15L^4 = 0$ means $L^2(1000 - L^2) = 0$. The critical points are 0 or $1000 = L^2$. The maximum is $\sqrt{1000}$. We have $Q''(L) = 300000L - 60L^3$ which is negative $-\sqrt{10}300000$ at $L = \sqrt{1000}$ so that this is a maximum.

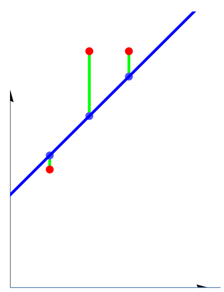
Problem 31.4: **Marginal revenue** f is the rate of change in **total revenue** F. As total and marginal cost, these are functions of the **cost** x. Assume the total revenue is $F(x) = -5x - x^5 + 9x^3$. Find the points, where the total revenue has a local maximum. Also here, use the second derivative test.

Solution:
$F' = -5 - 5x^4 + 27x^2 = 0$ gives the critical points $\pm\sqrt{27 \pm \sqrt{629}}$. Which is ± 0.4381 and ± 2.2821. The maximum is 2.2821 and a local maximum is -0.4381. Then, 0.43 is a local min as is -2.28 (global min). The second derivative F'' at the point is negative.

Problem 31.5: We do linear regression. Find the best line $y = mx + b$ through the points

$$(x_1, y_1) = (1, 3), (x_2, y_2) = (2, 6), (x_3, y_3) = (3, 6) .$$

To do so, first center the data by subtracting the average (2,5), then minimize $f(m) = \sum_{k=1}^{3} (mx_k - y_k)^2$ Now $m(x-2) + 5$ is the best fit.



Solution:
The centered data are $\{(-1, -2), (0, 1), (1, 1)\}$. The function is

$$f(m) = (2 - m)^2 + 1 + (m - 1)^2$$

The derivative is

$$f'(m) = -2(2 - m) + 2(-1 + m)$$

It is zero for $m = 3/2$. Plugging in $m(x - 2) + 5$ gives the regression line $y = 3/2x + 2$.

Oliver Knill, knill@math.harvard.edu, Math 1a, Spring, 2024