INTRODUCTION TO CALCULUS

MATH 1A

Unit 35: Review (since last review)

Substitution

Substitution replaces $\int f(x)\,dx$ with $\int g(u)\,du$ with $u = u(x), du = u'(x)dx$. Cases:

A) The integral of $f(x) = g(u(x))u'(x)$, is $G(u(x))$ where G is the anti-derivative of g. B) $\int f(ax+b)\,dx = F(ax+b)/a$, where F is the anti-derivative of f.

Examples:
A) $\int \sin(x^5)x^4\,dx = \int \sin(u)\,du/5 = -\cos(u)/5 + C = -\cos(x^5)/5 + C$.
B) $\int \log(x+7)\,dx = \int \log(u)\,du = (u\log(u) - u) + C = (x+7)\log(x+7) - (x+7) + C$.

Integration by parts

Use LIATE to decide what to differentiate.

A) Direct (use this if it can be done in one step)

$$\int \mathrm{x}\ \sin(x)\ dx = \mathrm{x}\ (-\cos(x)) - \int 1\ (-\cos(x)\ )\,dx = -x\cos(x) + \sin(x) + C\,dx\ .$$

B) Tic-Tac-Toe: for situations with an x^n in front. Example $x^2\sin(x)$:

x^2	$\sin(x)$	
$2x$	$-\cos(x)$	$\oplus$
2	$-\sin(x)$	$\ominus$
0	$\cos(x)$	$\oplus$

The anti-derivative is

$$-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C\ .$$

C) Merry go round: Example $I = \int \sin(x)e^x\,dx$. Use parts twice and solve for I.

Partial fractions

A) Either make a common denominator on the right hand side

$$\frac{1}{(x-a)(x-b)} = \frac{A(x-b)+B(x-a)}{(x-a)(x-b)}$$

and compare coefficients $1 = Ax - Ab + Bx - Ba$ to get $A + B = 0, Ab - Ba = 1$ and solve for A, B.

B) Use the residue method: $f(x) = p(x)/(x-a)(x-b)$ with $a \neq b$, the coefficients A, B in

$$\frac{p(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

can be obtained as $A = p(a)/(a-b)$ and $B = p(b)/(b-a)$.

Examples:
A) $\int \frac{1}{(x+1)(x+2)}\ dx = \int \frac{A}{x+1}\ dx + \int \frac{B}{x+2}\ dx$. Find A, B by multiplying out and comparing coefficients in the nominator.

B) Better: get A by plugging in $x = -2$ after multiplying with $x - 2$. Get B by plugging in $x = -1$ after multiplying with $x - 1$.

Checklists:

Integral techniques to consider

Try to crack the integral in the following order:

- ☐ Know the integral
- ☐ Substitution
- ☐ Integration by parts
- ☐ Partial fractions

Especially cool parts:

- ☐ Tic-Tac-Toe for integration by parts
- ☐ Hospital Method for partial fractions
- ☐ Merry go round method for parts

Integrals to know well

- ☐ $\sin(x)$
- ☐ $\cos(x)$
- ☐ $\tan(x)$
- ☐ $\log(x)$
- ☐ $\exp(x)$
- ☐ $1/x$

- [ ] $1/x^n$
- [ ] x^n
- [ ] $\sqrt{x}$
- [ ] $1/\cos^2(x)$
- [ ] $1/\sin^2(x)$
- [ ] $1/(1+x^2)$
- [ ] $1/(1-x^2)$
- [ ] $1/\sqrt{1-x^2}$

Applications

- [ ] **Average rate of change:** $[f(x+h)-f(x)]/h$
- [ ] **Derivative:** Limit of differences $[f(x+h)-f(x)]/h$ for $h \to 0$
- [ ] **Integral:** Limit of Riemann sums $S_n = [f(x_0)+f(x_1)+\cdots+f((n-1)x_k)]\Delta x$.
- [ ] **Newton step:** $T(x) = x - f(x)/f'(x)$.
- [ ] **Velocity:** Derivative of the position.
- [ ] **Acceleration:** Derivative of the velocity.
- [ ] **Concavity:** measured by $f''(x)$.
- [ ] **Marginal cost:** the derivative F' of the total cost F.
- [ ] **Average cost:** F/x where F is the total cost.
- [ ] **Probability distribution function:** non-negative, total $\int f(x)dx = 1$.
- [ ] **Cumulative distribution function:** anti-derivative of the PDF.
- [ ] **Expectation:** $\int xf(x)\,dx$, where f is the probability density function.
- [ ] **Moments:** $\int x^n f(x)\,dx$, where f is the probability density function.

Terminology

- [ ] **Operations research:** find extrema, critical points, derivative tests
- [ ] **Crisis management:** critical points and catastrophes.
- [ ] **Economics:** average, mar cost, marginal cost and total cost. Strawberry theorem.
- [ ] **Computer science:** sigmoid function $\sigma(x) = 1/(1+e^{-x})$, neural networks.
- [ ] **Statistics:** PDF, CDF, expectation, variance.
- [ ] **Distributions:** normal, geometric, exponential, Cauchy, arcsin, logistic.
- [ ] **Treasure hunting:** Bisection method, Newton method $T(x) = x - f(x)/f'(x)$.
- [ ] **Artificial intelligence:** sigmoid function, neural net functions, attention.
- [ ] **Music:** hull function, piano function.
- [ ] **Gastronomy:** wobbling table, bottle calibration.

Other notions

- [ ] **Hull function:** $\sin(x)\sin(10000x)$ has hull $|\sin(x)|$
- [ ] **Catastrophe:** A parameter c at which a local minimum disappears.
- [ ] **Artificial intelligence:** Sigmoid σ, neural network $c\sigma(ax+b)$
- [ ] **Entropy:** $-\int f(x)\log(f(x))\,dx$.

- **Bottles:** How to calibrate bottles. The calibration formula.
- **Wobbly chair:** One can turn a chair on any lawn to stop it from wobbling.
- **Derivative rule:** The hit: "low d high take high d low, cross the line and square the low"
- **Midi function:** $f(m) = 440 * 2^{(m-69)/12}$.

Core concepts:

- **Algebra:** Algebra, Power, Log
- **Fundamental:** The fundamental theorem of calculus
- **Trigonometry:** Fundamental theorem of trigonometry
- **Intermediate:** Intermediate value theorem
- **Mean value:** Mean value theorem
- **Extrema:** First derivative test
- **Extrema:** Second derivative test
- **Derivatives:** slope rate of change
- **Integrals:** area, volume
- **Limits:** Definitions and Hospital!
- **Limits:** Squeeze theorem
- **Continuity:** know the enemies of continuity
- **Numerics:** Riemann sums
- **Rules:** Differentiation and integration rules.
- **Methods:** Integration by parts, Substitution, Partial fraction.

People:

- **Newton, Leibniz**: Fundamental theorem, Newton method
- **Hospital, Bernoulli**: Hospital method
- **Riemann**: Riemann sum
- **Bolzano**: Intermediate and mean value theorem
- **Rolle**: Rolle's theorem
- **Fermat**: Fermat principle
- **Gini**: Gini Coefficient
- **Thom, Zeeman**: Catastrophes
- **Zeno**: Zeno paradox
- **Knuth**: Knuth notation
- **Bolt, Neyad**: Sports
- **Gibbs, Bolzmann, Shannon, Helmholz**: Entropy, Energy
- **Polya, Tao:** How to solve.
- **Whittaker-Shannon:** Interpolation.
- **Euler:** Gradus Suavitatis.
- **Gonnick:** Cartoons