INTRODUCTION TO CALCULUS

MATH 1A

Unit 4: Worksheet

Problem 1: Does the function $\frac{\cos(x)}{x}$ have a limit at $x \to 0$? If yes, what is it? If not, why does the limit not exist?

Solution:
No. As $\cos(x) \to 1$ (a finite value) and $1/x \to \infty$, there is no chance of a limit, when we take the product $\cos(x)/x$.

Problem 2: Does the function $\frac{\sin(x)}{e^x}$ have a limit at $x \to 0$? If yes, what is it? If not, why does the limit not exist?

Solution:
Yes. The limit is zero. Since $\sin(x) \to 0$ and $e^x \to 1$, we do not have a problem.

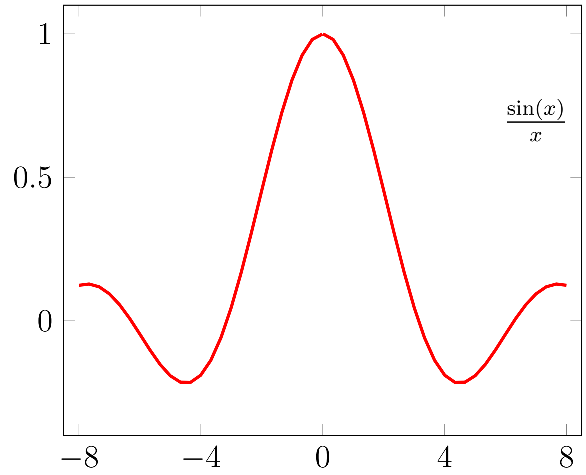
Problem 3): A prototype function for studying limits is the sinc function

$$f(x) = \frac{\sin(x)}{x} .$$

At which points can you be sure that the function has a limit? We will investigate the limiting behavior in class theoretically.

Solution:
At all points except $x = 0$. At $x = 0$ we have seen that the limit is 1.

Problem 4): First some experiment. Lets look at the graph of the function



If you look at the graph, does it appear that the function has a left or/and right limit everywhere?

Solution:
Yes, the graph looks smooth at $x = 0$. Whatever software we use, whether Desmos or Mathematica. There is appears to be no problem at $x = 0$.

Problem 5: Now that you know the answer to $\lim_{x \to 0} \sin(x)/x$, find the $\lim_{x \to 0} \frac{\sin(x^2)}{x^2}$.

Solution:
Yes, 1. Just treat $y = x^2$ as a new variable. Since $\lim_{y \to 0} \frac{\sin(y)}{y} \to 1$, we also have $\lim_{x \to 0} \frac{\sin(x^2)}{x^2} \to 1$.

Problem 6: Does the function $\frac{\sin(x^2)}{x}$ have a limit for $x \to 0$?

Solution:
Yes,0. We can write this as $x \frac{\sin(x^2)}{x^2}$. Since $x \to 0$ and $\frac{\sin(x^2)}{x^2} \to 1$, we have $x \frac{\sin(x^2)}{x^2} \to 0$.

Problem 7: Does the function $\frac{\sin(x)}{x^2}$ have a limit for $x \to 0$?

Solution:
No,it goes to infinity. We can see that as a product of $\mathrm{sinc}(x)$ and $1/x$. The first function has a limit, the second does not have a limit.

Problem 8: Does the function $\frac{x}{\sin(x)}$ have a limit for $x \to 0$?

Solution:
Yes,1. Since $\sin(x)/x \to 1$, we also have $x/\sin(x) \to 1$.

Problem 9: Does the function $\frac{\sin(x)}{|x|}$ have a limit for $x \to 0$?

Solution:
No,there is a jump. We can write this as $\frac{\sin(x)}{|x|} = \sin(x)/x(x/|x|)$. The next problem 10) shows that $x/|x| = \text{sign}(x)$ does not have a limit. So, also the product does not have a limit.

Problem 10: Does the function $\frac{x}{|x|}$ have a limit for $x \to 0$?

Solution:
No,there is a jump. This function $x/|x|$ is the $\text{sign}(x)$ function. It is 1 if $x > 0$ and -1 if $x < -0$. There is no limit therefore.

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