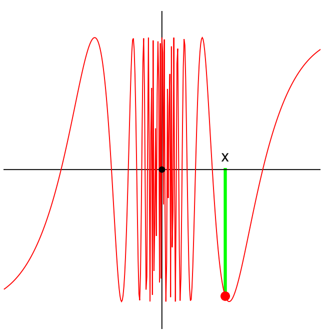
INTRODUCTION TO CALCULUS

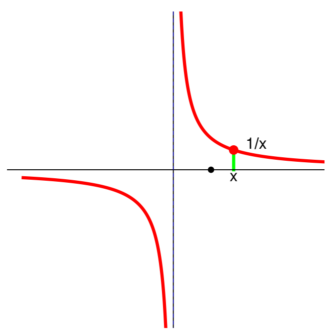
MATH 1A

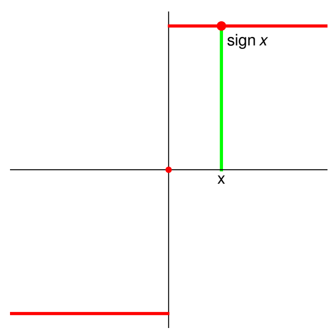
Unit 5: Worksheet

There are a few mechanisms for discontinuity . A function can **jump for good**, **badly rush to infinity** or have an **ugly oscillation**. All three cases come from a division by zero somewhere.

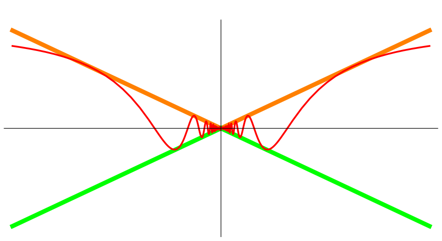
Nice Guys	**Good,Bad,Ugly Guys**
$x^2 + 4x + 6$	$1/x$ at 0
$\sin(x), \cos(x)$	$\tan(x)$ at $\pi/2$
$\exp(x)$	$\log\|x\|$ at 0
$\text{sinc}(x) = \frac{\sin(x)}{x}$	$\frac{1}{\cos(x)}$ at $\pi/2$







Squeeze theorem



Lets look at the **tamed devil** $f(x) = x\sin(\frac{1}{x})$. A bad oscillation is tamed by the **squeeze theorem**: We have $-|x| \leq x\sin(\frac{1}{x}) \leq |x|$ and both $g(x) = |x|$ and $h(x) = -|x|$ are continuous at 0 and have the same value at 0. Because f is sandwiched between two continuous functions which come together at the point, the function must be continuous and give that common value.

Which functions are continuous?

If you can assign a value at a point where it is not defined so that it becomes overall continuous, we just consider this continuous. We consider $(x^4 - 1)/(x - 1)$ to be continuous for example because it is equivalent for $x \neq 1$to $x^3 + x^2 + x + 1$ which is continuous. We consider $\sin(x)/x$ to be continuous because we can fill in a value 1 at $x = 0$ to make it continuous overall. We saw that $x \sin(x)$ is continuous everywhere because of the squeeze theorem. Which of the following functions are continuous?

Problem 1: $f(x) = x^2 + x^2 \sin(1/x^2)$

Solution:
Yes, the function is continuous with the unerstanding $f(0) = 0$. The only point to fix is $x = 0$. By the squeeze theorem, we can assign a value 0 at 0.

Problem 2: $f(x) = \sqrt{|x|}$

Solution:
This function is $\sqrt{x}$ for $x \geq 0$ which is continuous. The absolute value fixes it also for $x < 0$. The function is continuous overall.

Problem 3: $f(x) = |x^3|/x^3$

Solution:
This solution is equal to 1 for $x > 0$ and -1 for $x < 0$. It can not be fixed at $x = 0$. There is a jump discontinuity there. The function is not continuous.

Problem 4: $f(x) = |x|^2/x$

Solution:
Since $|x|^2 = x^2$, we can divide this by x and get x. This function is continuous. While the original expression $|x|^2/x$ was not defined at $x = 0$, we can find a value $f(0) = 0$ to render the function continuous.

Problem 5: $f(x) = \frac{1}{\sqrt{|x|}}$

Solution:
This function blows up at $x = 0$. There is no way, we can assign a finite value at $x = 0$. The function is not continuous.

Problem 6: $\frac{1}{\log|1/x|}$

Solution:
We see that at $x = 1$ we have $\log|1/x| = 0$ so that we divide by 0 there. The function is not continuous.

Problem 7: $\log(\log|x|)$

Solution:
This function is goes to ∞ both at $x = 0$ and $x = 1$. The function is actually not defined even for $x \in (-1, 1)$. It is not continuous on $[1, \infty)$ nor on $(-\infty, 1]$.

Problem 8: $1/(1 + |x|)$

Solution:
This function is continuous everywhere because $1 + |x|$ is continuous and positive everywhere.

Problem 9: $1/(1 - |x|)$

Solution:
This function has a pole at $x = 1$ and $x = -1$. It is not continuous.

Problem 10: $x^2/\sin(x)$

Solution:
This function can be written as $x(x/\sin(x))$. Both functions $f(x) = x$ and $g(x) = x/\sin(x)$ are continuous everywhere. So, also the product is at 0. However, there is a catch (which is a bit sneaky). The function is not continuous because at $x = \pi$ we have nothing to save us. The function is not continuous.

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