

# INTRODUCTION TO CALCULUS

MATH 1A

## UNIT 6: WORKSHEET

**Problem 1:** Take the definition of the derivative to find the derivative of the function  $f(x) = 2x$ .

**Solution:**

The answer is 2.

**Problem 2:** Take the derivative of  $f(x) = x^4$  by taking limits. We have to simplify

$$\frac{f(x+h) - f(x)}{h}.$$

Start with expanding  $(x+h)^4$ .

**Solution:**

We have to foil out  $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$ . Now,  $(x+h)^4 - x^4$  can be divided by  $h$  and gives  $4x^3 + h(\dots)$ . Setting  $h = 0$  gives  $4x^3$ .

**Problem 3:** Take the definition of the derivative to look at

$$\frac{f(x+h) - f(x)}{h}$$

in the case  $f(x) = |x|$ . What happens at  $x = 0$ ?

**Solution:**

The limit is not defined at 0 because the right limit 1 is not the same than the left limit  $-1$ .

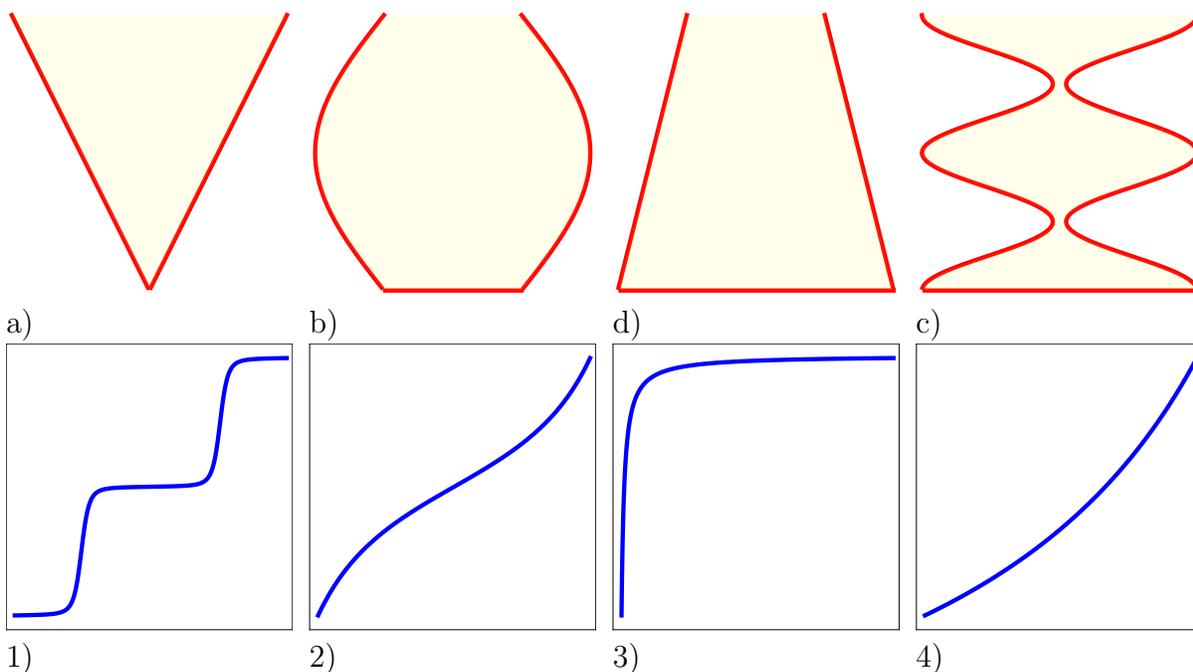
**Problem 4:** We start here with the task to find the derivative of  $f(x) = \ln(x)$  by taking limits. Simplify  $[\ln(x+h) - \ln(x)]$  as much as possible.

We will continue to work on this next week.

**Solution:**

We can simplify it to  $\ln((x + h)/x) = \ln(1 + h/x)$ . We will see that  $\lim_{h \rightarrow 0} \ln(1 + h/x)/h$  is  $1/x$ . This shows that the derivative of  $\ln$  is  $1/x$ .

**Problem 5:** In the famous **bottle calibration problem**, we fill a circular bottle or glass with constant amount of fluid. Plot the height of the fluid in the bottle at time  $t$ . Assume the radius of the bottle is  $f(z)$  at height  $z$ . Can you find a formula for the height  $g(t)$  of the water? This is not so easy. But we can find the rate of change  $g'(t)$ . Assume for example that  $f$  is constant, then the rate of change is constant and the height of the water increases linearly like  $g(t) = t$ . If the bottle gets wider, then the height of the water increases slower. There is definitely a relation between the rate of change of  $g$  and  $f$ . Before we look at this more closely, let's try to match the following cases of bottles with the graphs of the functions  $g$  qualitatively. In each of the bottles, we call  $g$  the height of the water level at time  $t$ , when filling the bottle with a constant stream of water. Can you match each bottle with the right height function.



**Solution:**

The key in this problem is to look at  $g'(t)$ , the rate of change of the height function. Because  $[g(t+h) - g(t)]$  times the area  $\pi f^2$  is a constant times the time difference  $h = dt$ , we have **bottle calibration formula**

$$g' = \frac{1}{\pi f^2}.$$

It relates the derivative function of  $g$  with the thickness  $f(t)$  of the bottle at height  $g$ . It tells that if the bottle radius  $f$  is large, then the water level increase  $g'$  is small and if the bottle radius  $f$  is small, then the liquid level change  $g'$  is large.