INTRODUCTION TO CALCULUS

MATH 1A

Unit 8: Worksheet

Problem 1: Get the slope of the tangent to the graph of $f(x) = xe^{-x}$ at $x = 0$.

Solution:
1

Problem 2: We know that $\frac{d}{dx}x^4$ is $4x^3$. (Was done in class the "old way" using a limit.) Derive this here from the product rule for $x^2 \cdot x^2$.

Solution:
$2xx^2 + x^2 2x = 4x^3$

Problem 3: Find the derivative of $1/x^3$ using the quotient rule.

Solution:
$x^3 * 0 - 3x^2/x^6 = -3/x^4$.

Problem 4: Find the derivative of the function $\cos(x)x$ at $x = 0$.

Solution:
$-\sin(x)x + \cos(x) = 1$ at $x = 0$.

Problem 5: If $f(x) = \sqrt{x}/x$, what is $f'(x)$? What is $f'(1)$?

Solution:
Using the quotient rule of course is crazy but we can do it $(x/(2\sqrt{x}) - \sqrt{x})/x^2 = -1/(2x^{3/2})$. Better of course is to use the rule for $f(x) = x^{-1/2}$.

Problem 6: Find the derivative of $1/e^x$ at $x = 1$.

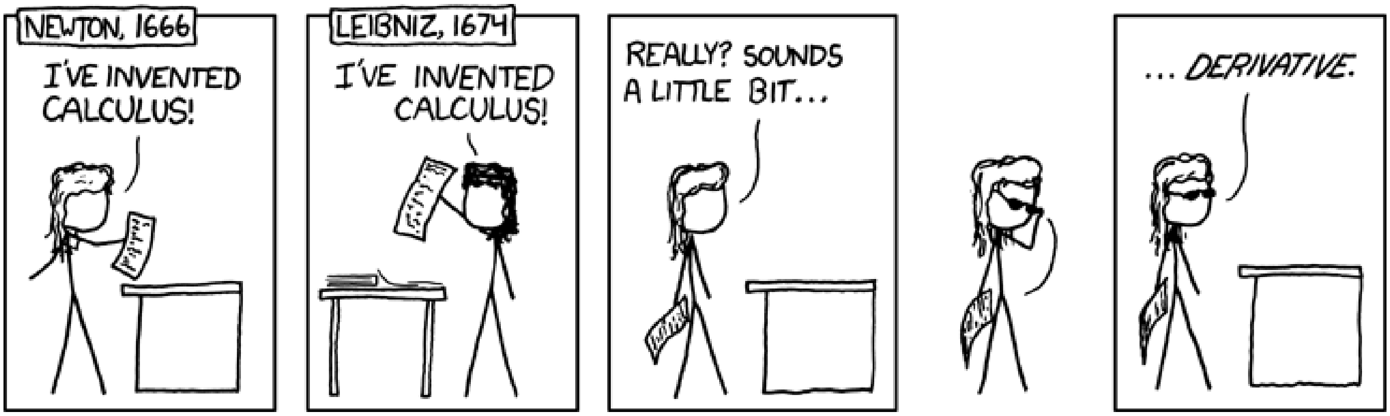
Solution:
The quotient rule gives $(e^x * 0 - e^x)/(e^{2x}) = -1/e^x$. Of course, simplifying first to e^{-x} and getting $-e^{-x}$ would be faster. We wanted to practice the quotient rule however.

Problem 7: Remember the formula $\sin(2x) = 2\sin(x)\cos(x)$? Differentiate both sides to get a formula for $\cos(2x)$.

Solution:
Differentiation to the left gives $2\cos(2x)$. To the right, we have $2\cos^2(x) - 2\sin^2(x)$. This gives a nice identity $\cos(2x) = \cos^2(x) - \sin^2(x)$.

Problem 8: You are given $\arctan'(x) = 1/(1 + x^2)$. Find $\arctan''(x)$.

Solution:
Lets do it with the reciprocal rule: $-2x/(1 + x^2)^2$.



Source: XKCD

I.

NOVA METHODUS PRO MAXIMIS ET MINIMIS, ITEMQUE TANGENTIBUS, QUAE NEC FRACTAS NEC IRRATIONALES QUANTITATES MORATUR, ET SINGULARE PRO ILLIS CALCULI GENUS*).

Sit (fig. 111) axis AX, et curvae plures, ut VV, WW, YY, ZZ, quarum ordinatae ad axem normales, VX, WX, YX, ZX, quae vocentur respective v, w, y, x, et ipsa AX, abscissa ab axe, vocetur x. Tangentes sint VB, WC, YD, ZE, axi occurrentes respective in punctis B, C, D, E. Jam recta aliqua pro arbitrio assumta vocetur dx, et recta, quae sit ad dx, ut v (vel w, vel y, vel z) est ad XB (vel XC, vel XD, vel XE) vocetur dv (vel dw, vel dy, vel dz) sive differentia ipsarum v (vel ipsarum w, vel y, vel z). His positis, calculi regulae erunt tales.

Sit a quantitas data constans, erit da aequalis 0, et $d\overline{ax}$ erit aequalis adx. Si sit y aequ. v (seu ordinata quaevis curvae YY aequalis cuivis ordinatae respondenti curvae VV) erit dy aequ. dv. Jam *Additio et Subtractio*: si sit z — y + w + x aequ. v, erit $d\overline{z - y + w + x}$ seu dv aequ. dz — dy + dw + dx. *Multiplicatio*: $d\overline{xv}$ aequ. xdv + vdx, seu posito y aequ. xv, fiet dy aequ. xdv + vdx. In arbitrio enim est vel formulam, ut xv, vel compendio pro ea literam, ut y, adhibere. Notandum, et x et dx eodem modo in hoc calculo tractari, ut y et dy, vel aliam literam indeterminatam cum sua differentiali. Notandum etiam, non dari semper regressum a differentiali Aequatione, nisi cum quadam cautione, de quo alibi. Porro *Divisio*: $d\frac{v}{y}$ vel (posito z aequ. $\frac{v}{y}$) dz aequ. $\frac{\pm v dy \mp y dv}{yy}$.

Quoad *Signa* hoc probe notandum, cum in calculo pro litera substituitur simpliciter ejus differentialis, servari quidem eadem signa, et pro + z scribi + dz, pro — z scribi — dz, ut ex addi-

*) Act. Erud. Lips. an. 1684.

Leibniz 1684 paper in which the product and quotient rule is introduced.

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