INTRODUCTION TO CALCULUS

MATH 1A

Unit 15: Worksheet

Exactly 20 years after the iconic "Mean girls" movie with Lindsay Lohan as the high school student, an other "Mean girls" movie has hit the theaters. The story is closer to a musical but very similar to the original movie. Even the limit question is the same! Here is the original question

Problem 1: Find the limit

$$\lim_{x\to 0} \frac{\ln(1-x)-\sin(x)}{1-\cos^2(x)} .$$

Solution:
It is an indefinite form $0/0$. Bring it to the hospital. We have $\lim_{x\to 0}[-1/(1-x)-\cos(x)]/(-2\cos(x)\sin(x))$. But note that $x=0$ is no indefinite form $0/0$ any more. It is -2/0. The limit does not exist.

As a director of the movie, I might actually have changed the limit a bit and made it more interesting. Can you solve the following limit problem?

Problem 2: Find the limit

$$\lim_{x\to 0} \frac{\ln(1-x)+\sin(x)}{1-\cos^2(x)} .$$

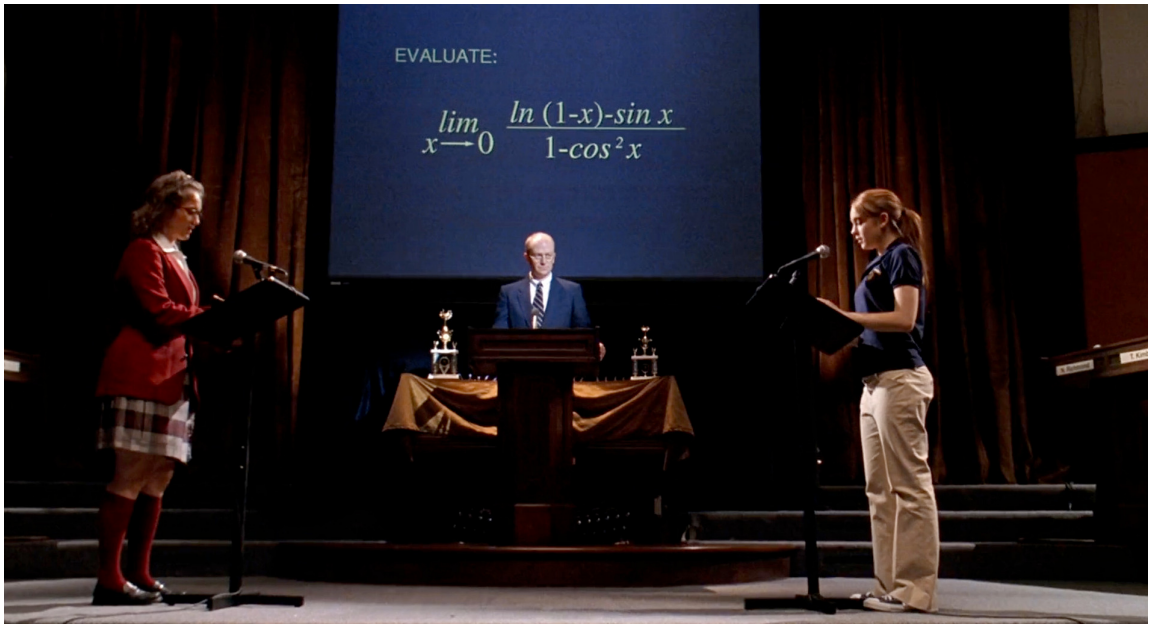
Solution:
Hospital gives $\lim_{x\to 0}[-1/(1-x)+\cos(x)]/(-2\cos(x)\sin(x))$. This is again an indefinite form. Bring it again to the hospital. The limit is $-1/2$.

Problem 3: In the movie, the contestants did not have access to paper or pencil and had to do things in their head. Can you imagine a strategy which without Hospital could solve both problems faster? There is a strategy (at least in 1).

Solution:
One way is to use linearization. $\ln(1-x)$ can be replaced by $-x$ and $\sin(x)$ by x. The top therefore looks like $-2x$. The bottom is $\sin^2(x)$ which is like x^2. In problem 1), we therefore have the problem to find the limit $(-2x)/x^2$ which does not have a limit. In the modified problem, we can not use linearization but would have to replace $\ln(1-x)$ with $-x-x^2/2$, the best quadratic approximation. Now we deal with the limit $(-x^2/2)/x^2 = \boxed{-1/2}$.
Here is an other strategy for 2): replace $\sin(x)$ with x and $1-\cos^2(x) = \sin^2(x)$ with x^2. The problem reads then

$$\lim_{x \to 0} \frac{\ln(1-x)+x}{x^2} .$$

This expression can be brought to the hospital twice and the computations are not that hard. The first time, we get $[-1/(1-x)+x]/(2x)$. The second visit to the Hospital gives $-1/(1-x)^2/2$ which gives $\boxed{-1/2}$.



From the original "Mean Girls 2004" movie. The 2024 movie borrowed the same slide.

From the original Mean Girls 2004 movie. On the board, you see a topic we cover next week in this class.

From the new "Mean Girls 2024" movie. Tina Fey is again the math teacher. The board shows a "healing limit problem" and a problem, where no healing is necessary.

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