Unit 20: Worksheet

The intermediate value theorem

Problem 1: Today, the average temperature is $48°$ Fahrenheit. Yesterday, it had been $58°$. Is it true that there was a moment last night, where the temperature had been exactly 50 degree Fahrenheit.

Solution:
let $T(t)$ be the temperature. Define $f(t) = T(t) - 50$. Now, Let $t = 0$ be the time yesterday and $t = 1$ be the time today. Then $f(0) = 48 - 50 < 0$ and $f(1) = 58 - 50 = 8 > 0$. The function $T(t)$ and so $f(t)$ is continuous. Therefore, $f(t)$ has a root in $[0, 1]$ by the intermediate value theorem.

Problem 2: Argue why there was a time in your life whether you were 1000 times longer than your average teeth length.

Solution:
Let $f(t) = 1000T(t) - L(t)$, where $L(t)$ is your length and $T(t)$ is the teeth length in cm. Now $T(0) = 0, L(0) > 0$ so that $f(0) < 0$. Now at $t = 1$, grown up, we have $T(1) = 2$ (maybe, but definitely less than 4) and $L(1) = 170$ (definitely not more than 300). So, $f(1) = 2000 - 170 > 0$. This is clear even if you would have taken 5mm teeth now and 3 Meter size. Since both T(t) and L(t) are continuous functions, also f(t) is continuous. The intermediate value theorem assures that f has a root.

Problem 3: How would you find a root of the function $f(x) = \cos(x) - x$ using a calculator and without taking derivatives.

Solution:
Check $f(0) = 1$ and $f(\pi/2) = -\pi/2 < 0$. There is a root between 0 and $\pi/2$. Now evaluate $f(\pi/4)$ and see whether it is positive or negative. This decides whether the root is in $[0, \pi/4$ or in $[\pi/4, \pi/2]$. With a calculator, we see $f(\pi/4) = 2/\sqrt{2} - \pi/4 = -0.078 < 0$. The root is therefore in $[\pi/4, \pi/2]$.

Problem 4: Is there a point x, where

$$\frac{1}{\sin(x)} = \frac{1}{2} \ ?$$

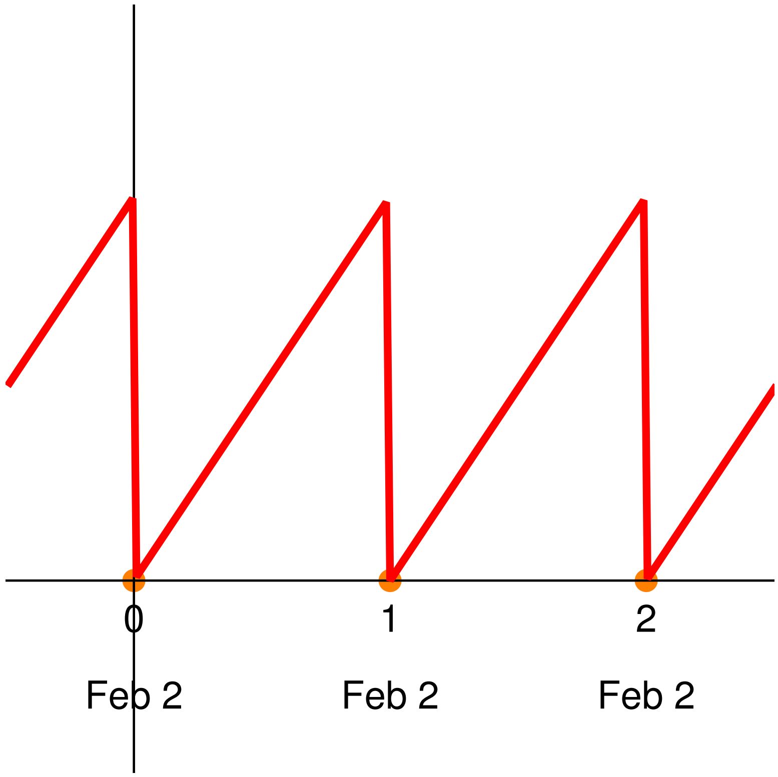
We have $1/\sin(\pi/2) = 1$ and $1/\sin(3\pi/2) = -1$.

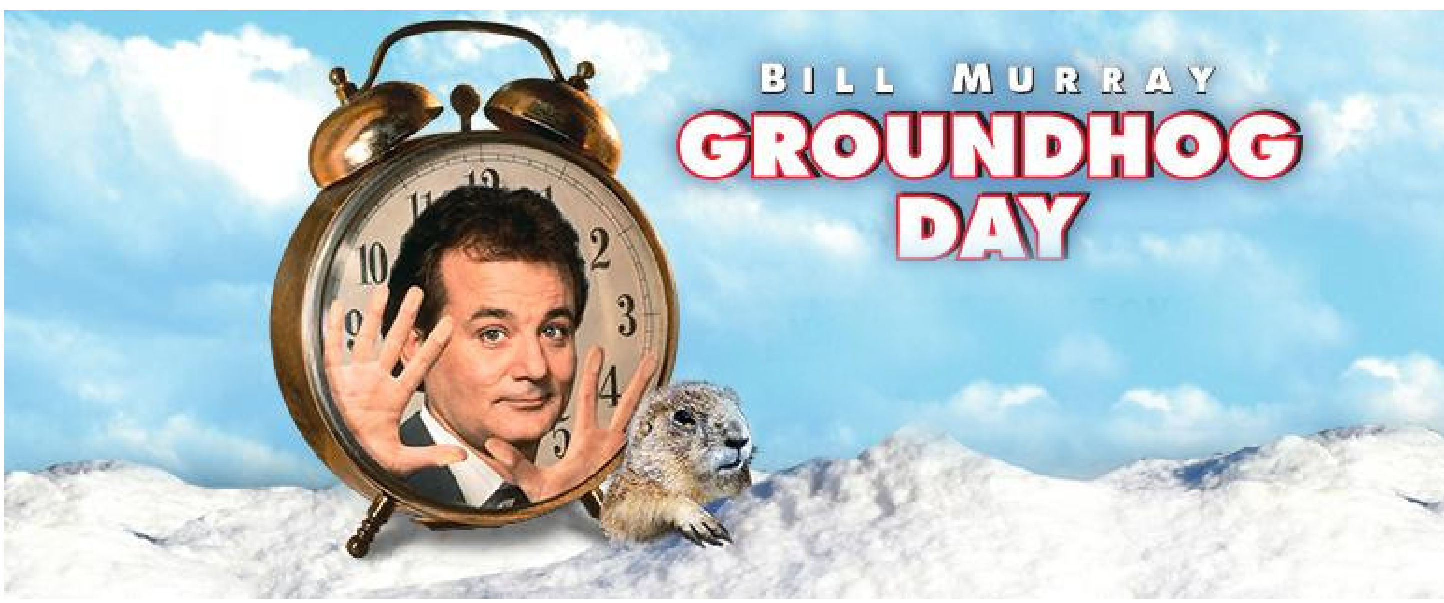
Solution:
The function $1/\sin(x)$ is not continuous. While we know points where f is negative and points where f is positive, we do not have points where $f = 0$ or $f = 1/\sqrt{2}$.

Problem 5: The earth's diameter is 12'756 km in average. Is there a point on earth where the distance to its anti-pod is exactly 12'756 km?

Solution:
Take the function $f(x) = d(x) - 12756$. We know that the average of f is 0. It is not possible that the function is everywhere positive or negative. So, either the function is everywhere 0 or negative somewhere and positive somewhere else.

Problem 6: The function $g(x) = x - \text{floor}(x)$ is a **ground hog function**. If you know the movie with **Bill Murray**, you know why. We know $g(0.9) = 0.9$ and $g(1.1) = 0.1$. Can you conclude that there is a point between 0.9 and 1.1 where $g(x) = 0.5$? What does the intermediate value theorem tells here?





Solution:
We are looking at roots of the function $f(x) = g(x) - 0.5$. We know $f(0.9) = 0.4$ and $f(1.1) = -0.4$. But f is not continuous at integers. We can not conclude that there is a value where f is 0.

Oliver Knill, knill@math.harvard.edu, Math 1a, Spring, 2024