Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2024

2/28/2024: First Hourly Practice B

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

Please write neatly. Use the same page for the answer if possible.

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		100

Problem 1) TF questions (10 points) No justifications are needed.

1) **T** F — The function $f(x) = \exp(-x^2) - 1$ has the root $x = 0$.

Solution:
The exp function is 1 at $x = 0$.

2) **T** F — $\ln(\ln(e)) = 0$.

Solution:
Yes, $\log(e) = 1$ and $\log(1) = 0$.

3) T **F** — The function $f(x) = x^2/(1-x^2)$ is continuous everywhere on the real axes.

Solution:
It has a pole at $x = 1$ and $x = -1$ and is not continuous there.

4) T **F** — $f'(x) = 0$ and $f'''(0) < 0$ at $x = 0$ assures that f has a maximum at $x = 0$.

Solution:
It is the second derivative test, not the third one

5) T **F** — A continuous function on $(-1, 1)$ has a global maximum and a global minimum.

Solution:
Bolzano's extremal value theorem assumes that the interval is closed. A countner example is 1/((1-x)(1+x)) which is continuous on $(-1, 1)$ but not on $[-1, 1]$.

6) T **F** If f and g are differentiable then f/g is differentiable

Solution:
Almost always, but it is possible that $g = 0$

7) T **F** The Groundhog function $\arctan(\tan(x))$ is continuous everywhere

Solution:
It is only piecewise continuous. There are jumps at $x = \pi/2 + k\pi$.

8) T **F** $\lim_{x \to 0} x^x = 0$.

Solution:
We have seen this in a homework. The limit is 1 because $x \ln(x) \to 0$ and $x^x = e^{x \ln(x)}$.

9) **T** F $\frac{d}{dx} e^x/x = \frac{e^x x - e^x}{x^2}$.

Solution:
Yes, the quotient rule has been used correctly.

10) T F The derivative of $h(x) = f(x) * g(x)$ is $h'(x) = f'(x)g'(x)$.

Solution:
False formula for the product rule!

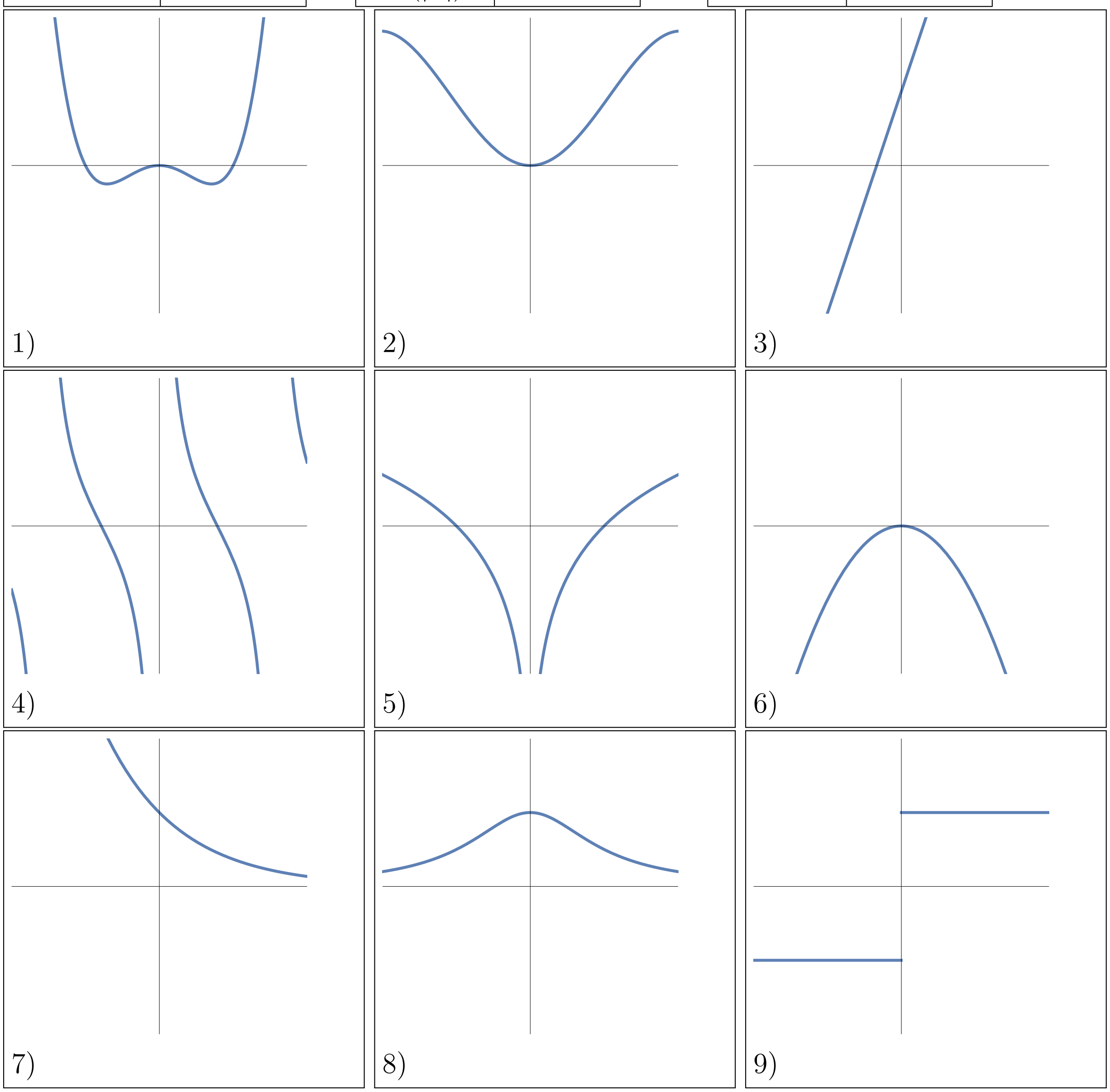
Problem 2) Analysis (10 points) No justifications are needed.

Match the functions with the graphs.

Function	Enter 1-9
$1/(1+x^2)$	
$\cot(2x)$	
$3x+1$	

Function	Enter 1-9
$x\sin(x)$	
$\exp(-x)$	
$\ln(\|x\|)$	

Function	Enter 1-9
$\text{sign}(x)$	
x^4-x^2	
$-x^2$	

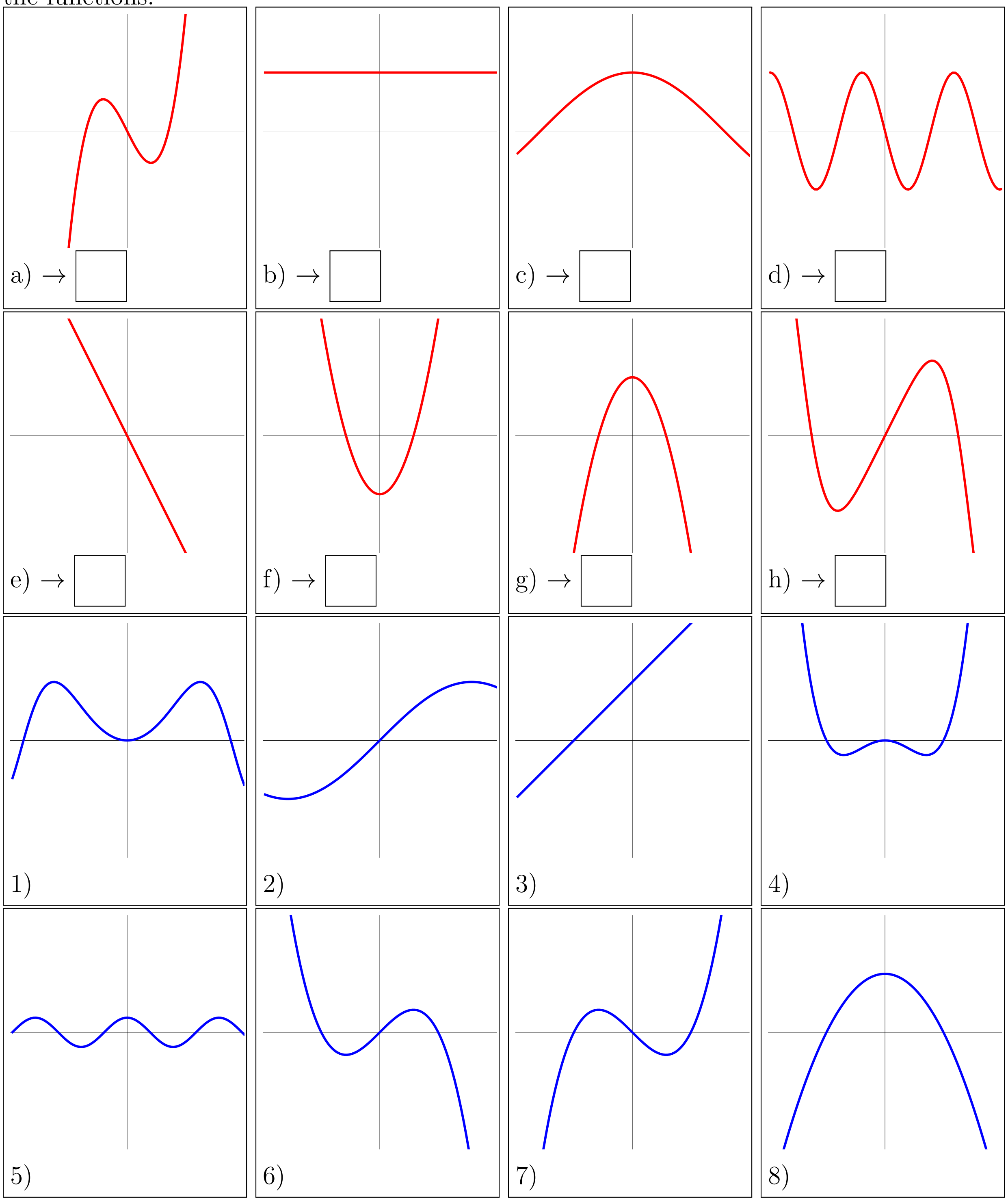


Solution:

8)	2)	9)
4)	7)	1)
3)	5)	6)

Problem 3) Graphing (10 points) No justifications are needed.

In the first pictures, we see the first derivatives f'. Match them with the functions f in 1-8. Note that the functions above are the derivative and the functions below are the functions.



Solution:
4325
8761

Problem 4) Continuity (10 points)

Each of the following functions has a point x_0, where the function is not defined. Find the limit $\lim_{x \to x_0} f(x)$ or state that the limit does not exist.

a) (2 points) $f(x) = \frac{1-2x^3}{1-x}$, at $x_0 = 1$.

b) (2 points) $f(x) = \sin(\sin(5x))/\sin(7x)$, at $x_0 = 0$.

c) (2 points) $f(x) = \frac{\exp(-3x)-1}{\exp(2x)-1}$, at $x_0 = 0$.

d) (2 points) $f(x) = \frac{2x}{\ln|x|}$, at $x_0 = 0$.

e) (2 points) $f(x) = \frac{(x-1)^{10}}{(x+1)^{10}}$, at $x_0 = -1$.

Solution:
a) No Limit.
b) Use Hospital: $5/7$.
c) Use Hospital: $-3/2$
d) Use Hospital: 0
e) No limit.

(*) As usual, we extend continuity to functions for which a continuation is possible to initially not defined points, like $f(x) = (x^2 - 1)/(x-1)$, which is considered continuous everywhere because we can fill in a function value for $x = 1$ which makes it continuous.

Problem 5) Derivatives (10 points)

Find the derivatives of the following functions. If you use a differentiation rule, note which one you use.

a) (2 points) $f(x) = \sqrt{\ln(x+1)}$.

b) (3 points) $f(x) = 7\sin(x^3) + \frac{\ln(5x)}{x}$.

c) (3 points) $f(x) = \ln(\sqrt{x}) + \arctan(x^3)$.

d) (2 points) $f(x) = e^{5\sqrt{x}} + \tan(x)$.

Solution:
a) $1/[2(1+x)\sqrt{\ln(1+x)}]$
b) $21x^2\cos(x^3) + [1 - \ln(5x)]/x^2$. c) $1/(2x) + 3x^2/(1+x^6)$
d) $\frac{5}{2\sqrt{x}}e^{5\sqrt{x}} + 1/\cos^2(x)$

Problem 6) Limits (10 points)

Find the limits $\lim_{x \to 0} f(x)$ for the following functions:

a) (2 points) $f(x) = \frac{\exp(3x)-\exp(-3x)}{\exp(5x)-\exp(-5x)}$.

b) (3 points) $f(x) = \frac{\cos(3x)-1}{\sin^2(x)}$.

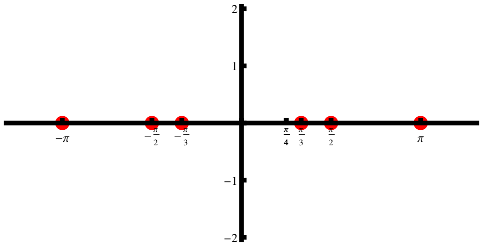
c) (3 points) $f(x) = [\arctan(x) - \arctan(0)]/x$.

d) (2 points) $f(x) = \frac{\ln(7x)}{\ln(11x)}$.

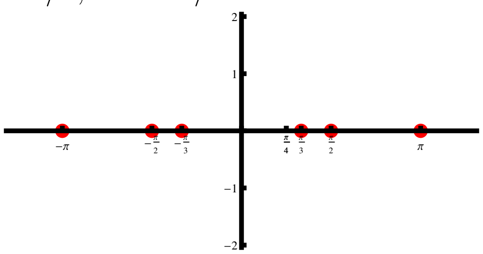
Solution:
a) Use Hospital: $\frac{3\exp(3x)+3\exp(-3x)}{5\exp(5x)+5\exp(-5x)}$. Now take the limits: 3/5.
b) Use Hospital once $-3\sin(3x)/(2\sin(x)\cos(x))$ and then a second time $9\cos(3x)/2\cos^2(x) - 2\sin^2(x) \to -9/2$
c) This is just the derivative of arctan at 0 which is 1. Hospital gives the same.
d) Use Hospital using $d/dx \ln(ax) = 1/x$ for all constants a and get 1.

Problem 7) Functions (10 points) no explanations needed

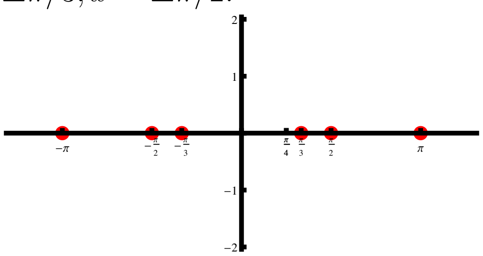
a) Draw the sin function and mark the values of $\sin(x)$ at $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$.



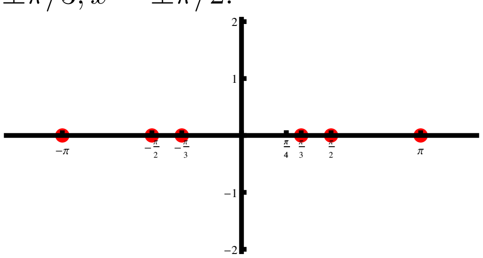
b) Draw the cos function and mark the values of $\cos(x)$ at $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$.



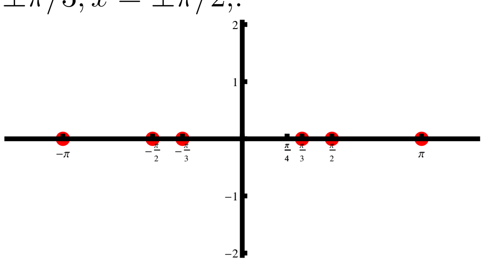
c) Draw the tan function and mark the values of $\tan(x)$ at $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$.



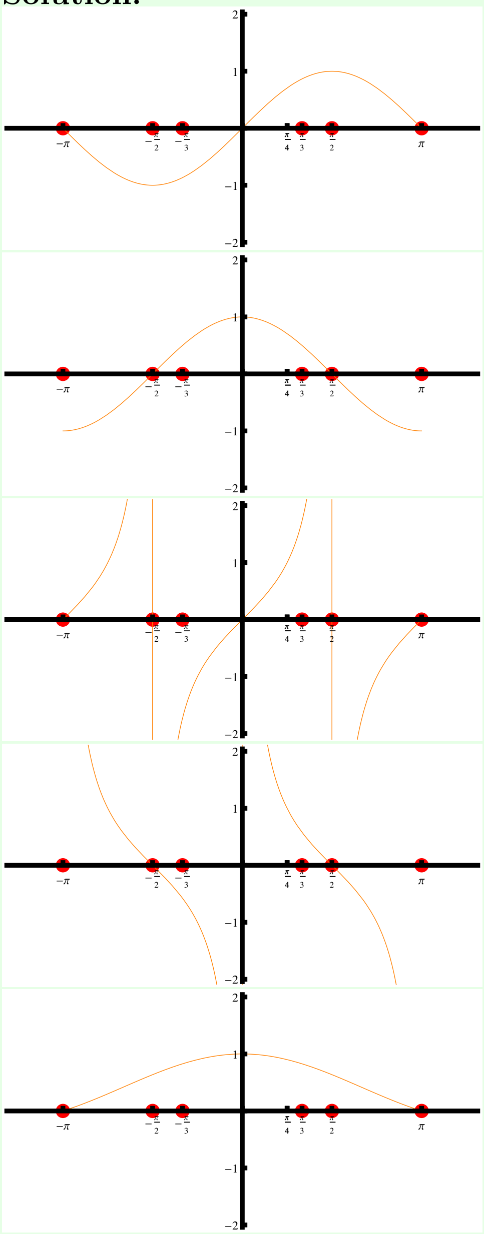
d) Draw the cot function and mark the values of $\cot(x)$ at $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$.



e) Draw the sinc function $f(x) = \sin(x)/x$ and mark the points $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2,$.



Solution:



Problem 8) Extrema (10 points)

You got a batch of strong **Neodym magnets**. They are ring shaped. Assume the inner radius is x, the outer radius y is 1 and the height is $h = x$, we want to maximize the surface area $A = 2\pi(y-x)h + 2\pi(y^2 - x^2)$. This amount of maximizing

$$f(x) = 2\pi(1-x)x + 2\pi(1-x^2)$$

a) (2 points) Using that $f(x)$ is a surface area, on what interval $[a, b]$ needs f to be considered?

b) (3 points) Find the local maxima of f inside the interval.

c) (3 points) Use the second derivative test to verify it is a maximum.

d) (2 points) Find the global maximum on the interval.

Solution:
a) The surface area is non-negative on $[0, 1]$.
b) $f'(x) = 2\pi(1-4x)$. It is zero at $x = 1/4$. This is the only critical point.
c) $f''(x) = -8\pi$ is negative. By the second derivative test, the critical point is a local maximum.
d) To find the global maximum, compare $f(0) = 2\pi = 6.28..$, $f(1/4) = 2\pi/16 + 2\pi(3/4) = 7.07..$ and $f(1) = 0$. Since the graph of f is quadratic and is everywhere concave down, one could see also that $1/4$ is a global maximum also without evaluating the end points.

Problem 9) Algebra (10 points)

Simplify the following terms. log denotes the natural log and $\log_{10}$ the log to the base 10. Each result in a)-c) is an integer or a fraction

a) (2 points) $\exp(\log(2)) + e^{3\log(2)}$

b) (2 points) $\log(1/e) + \exp(\log(2)) + \log(\exp(3))$.

c) (2 points) $\log_{10}(1/100) + \log_{10}(10000)$

d) (4 points) Simplify

$$\cos(\arccos(x)) \ .$$

to get the derivative of $\cos(\arccos(x))$.

Solution:
a) $\boxed{10}$
b) -1+2+3= $\boxed{4}$
c) -2+4 = $\boxed{2}$
d) $\cos(\arccos(x)) = x$ has the derivative 1.

Problem 10) Linearization (10 points, 5 points each)

a) (5 points) Use linearization to estimate $\sqrt[5]{1.001}$.

b) (5 points) Use linearization to estimate $e^{\ln(2)+0.01}$.

Solution:
a) Take the function $f(x) = \sqrt[5]{x} = x^{1/5}$. It has the derivative $(1/5)x^{-4/5}$. At the point 1 this is $1/5$. We therefore have the linearization $f(1) + f'(1)0.001 = 1 + 0.001/5 = 1.0002$.
b) We deal with the function $f(x) = e^x$ and use the point $a = \ln(2)$. We have $f(a) = e^{\ln(2)} = 2$ and $f'(a) = e^a = 2$. So, the linearization is $2 + 2 \cdot 0.01 = 2.02$. The actual answer is 2.0201.