Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2024

4/3/2024: Second Hourly Practice B

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

Please write neatly. Use the same page for the answer if possible.

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		100

Problem 1) TF questions (10 points) No justifications are needed.

1) [T] [F] The chain rule assures that $\int e^{\sin(x)} \cos(x)\, dx = e^{\sin(x)} + C$

Solution:
Yes, because the derivative of the right hand side is the integrand.

2) [T] [F] A Riemann sum approximation S_n of $S = \int_0^1 x^5\, dx$ satisfies $|S - S_n| \leq 5/n$.

Solution:
We have seen that the error is smaller or equal than M/n, where M is the maximal size of the derivative of f on the interval. In this case it is maximally 5.

3) [T] [F] We have $\int_0^1 10f(x)\, dx = 10 \int_0^1 f(x)\, dx$.

Solution:
Yes, we can take constants out of the integral

4) [T] [F] $\int_0^1 \frac{1}{f(x)}\, dx = \ln(f(x))$.

Solution:
No, there is no such rule. Even so it looks tempting when looking at $f(x) = x$. Just try an example like $f(x) = 1$.

5) [T] [F] If $f(x) = 0$ everywhere, then $\int_0^1 f(x)\, dx = C$

Solution:
This is a definite integral. The answer is zero.

6) [T] [F] If $f \leq g$ then $\int_a^b g(x) - f(x)\, dx = \int_a^b g(x)\, dx - \int_a^b f(x)\, dx$.

Solution:
We have seen this in a homework.

7) [T] [F] $\int_a^b \frac{d}{dx} f(x)\, dx = \frac{d}{dx} \int_a^b f(x)\, dx$.

Solution:
The left hand side is $f(b) - f(a)$, the right hand side is 0.

8) [T] [F] If a parameter c defining a function changes, then it is possible that the location of the smallest equilibrium changes discontinuously.

Solution:
yes, this is why we call this a catastrophe.

9) [T] [F] If T is a Newton step, then $T(x) = x - f'(x)/f(x)$.

Solution:
It is opposite. $T(x) = x - f(x)/f'(x)$.

10) [T] [F] If f is a continuous function such that $f(-1) = f(1) = 1$ and $f(0) = -1$, then f has at least two different roots.

Solution:
This follows from the intermediate value theorem.

Problem 2) Theorems (10 points) No justifications are needed.

---

a) Formulate the **fundamental theorem of calculus** for definite integrals.

---

b) Formulate the **chain rule**.

---

c) Formulate the **intermediate value theorem**.

---

d) Formulate the **mean value theorem**.

---

e) Formulate **Rolle's theorem**.

Solution:
a) $\int_a^b f'(x)\,dx = f(b) - f(a)$.
b) $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$.
c) If f is continuous and the sign of $f(a)$ and $f(b)$ is different, then there exists $x \in (a, b)$ such that $f(x) = 0$.
d) If f is differentiable on $[a, b]$, then there exists a point in (a, b) such that $f'(x) = [f(b) - f(a)]/(b - a)$. Instantaneous and average rate of change can be the same when choosing the midpoint correctly.
e) If f is differentiable on $[a, b]$ and $f(b) = f(a)$, then f has a critical point in (a, b). This is a special case of the mean value theorem.

Problem 3) Matching (10 points)

a) 5 (points) Which of the following integrals are zero?

Integral	Check if zero
$\int_{-2\pi}^{2\pi} \sin(x)\, dx$	
$\int_{-2\pi}^{2\pi} e^x - e^{-x}\, dx$	
$\int_{-2\pi}^{2\pi} \frac{1}{1+x^2}\, dx$	
$\int_{-2\pi}^{2\pi} x^2 - \frac{1}{x^2}\, dx$	
$\int_{-2\pi}^{2\pi} \lvert\sin(x)\rvert\, dx\, dx$	

Solution:
The first two are zero, the last three are not.

b) (5 points) For which of the following functions can you apply the mean value theorem MVT for all $[a, b]$.

Function	Check if MVT is applicable
$\lvert\cos(x)\rvert$	
e^x	
$1/x$	
$\arctan(x)$	
$\tan(x)$	

Solution:
Only for the $\arctan(x)$ and e^x because these are the only functions that are differentiable everywhere.

Problem 4) Chain rule (10 points)

Find the derivatives of the following functions:

a) $e^{2e^{3x}}$

b) $\ln(2\ln(3x))$

c) $\cos(2\sin(3x))$

d) $\arctan(2e^{3x})$

e) $\tan(2\tan(3x))$

Solution:
a) $e^{2e^{3x}}2e^{3x}3 = 6e^{3x+2e^{3x}}$.
b) Use what we have seen alread that $d/dx\ln(cx) = 1/x$. The constant goes away. We end up with $\frac{1}{x\log(3x)}$
c) $-\sin(2\sin(3x))2\cos(3x)3$.
d) $\frac{1}{(1+(2e^{3x})^2)}6e^{3x}$.
e) $\frac{1}{\cos^2(2\tan(3x))}6/\cos^2(3x)$.

Problem 5) Related rates (10 points)

Assume you are on a roller coaster $y = 2\pi x + \sin(xy)$ and $x' = 3$. What is y' at the point $x = 1$, $y = \pi$?

Solution:
$y' = 2\pi x' + \cos(xy)[x'y + xy']$. Now fill in the data to get $y' = 2\pi 3 + \cos(\pi)[3\pi + y'] = 2\pi 3 - 3\pi - y'$ so that $2y' = 3\pi$. So, $y' = 3\pi/2$.

Problem 6) Implicit Differentiation (10 points)

Assume that y is a function of x and $x^4 + y^4 = x^2 + y^2 - y + 1$. We can not solve the equation for y and differentiate. But we can compute the derivative y' anyway. Do this at $x = 1, y = 1$.

Solution:
Differentiate to get $4x^3 + 4y^3y' = 2x + 2yy' - y'$. Plug in $x = 1, y = 1$ to get $4 + 4y' = 2 + 2y' - y' = 0$. So $3y' = -2$. So $y' = -2/3$.

Problem 7) Definite integrals (10 points)

Evaluate the following definite integrals.

a) $\int_1^2 \frac{1}{x} + \frac{1}{x^2}\, dx$

b) $\int_1^2 \frac{3}{1+x^2}\, dx$

c) $\int_1^2 3\sin(5x)\, dx$

d) $\int_1^2 e^{4x}\, dx$

e) $\int_1^2 \tan(x) dx$

Solution:
a) $\ln(x) - \frac{1}{x}|_1^2 = \ln(2) + 1/2$.
b) $3\arctan(x)|_1^2 = 3\arctan(2) - 3\arctan(1)$.
c) $-3\cos(5x)/5|_1^2 = 3\cos(5)/5 - 3\cos(10)/5$.
d) $e^{4x}/4|_1^2 = e^8/4 - e^4/4$.
e) $-\ln(\cos(x))|_1^2 = -\ln(\cos(2)) + \ln(\cos(1))$ (This by the way was an improper integral. The function $\tan(x)$ has a singularity in the interval $[1, 2]$. The above expression still makes sense when interpreted in an appropriate way.)

Problem 8) Anti derivatives (10 points)

Solve the indefinite integrals.

a) $\int \frac{1}{1-x^2}\, dx$
Hint: write this as $[1/(1-x) + 1/(1+x)]/2$.

b) $\int \frac{\sin(x)}{\cos^2(x)}\, dx$

c) $\int \cot(x)\, dx$

d) $\int x^5 - \frac{1}{x^5}\, dx$

e) $\int \frac{2x}{1+x^2}\, dx$

Solution:
a) $[-\ln(1-x) + \ln(1+x)]/2 + C$
b) $-1/\cos(x) + C$
c) $\ln(\sin(x)) + C$
d) $x^6/6 + 1/(4x^4) + C$
e) $\ln(1+x^2) + C$

Problem 9) Newton Step (10 points)

a) Do a Newton step $T(x) = x - f(x)/f'(x)$ to find the root of $\cos(\pi x) - x$ starting at $x_0 = 0$.

b) Now do a second Newton step.

Solution:
a) $T(0) = 1$.
b) $T(1) = -1$.

Problem 10) Catastrophes (10 points)

We look at the one-parameter family of functions $f_c(x) = 2x^3 + cx^2$, where c is a parameter.

a) (2 points) Find the critical points of $f_3(x)$.

b) (2 points) Find the critical points of $f_{-3}(x)$.

c) (2 points) Check that 0 is always a critical point.

d) (2 points) For which c is 0 a minimum?

e) (2 points) For which c does the catastrophe occur?

Solution:
a) $f'(x) = 6x^2 + 2cx = 2x(3x + c) = 0$ means either $x = 0$ or $x = -1$.
b) $f'(x) = 6x^2 - 6x = 0$ means either $x = 0$ or $x = 1$.
c) $f'(x) = 5x^2 - 2cx$ is always 0 independent of c.
d) $f''(0) = 2c = 2c$. We see that for $c > 0$ we have a minimum and for $c < 0$ a maximum.
e) At $c = 0$ the nature of the critical point changes. The parameter $c = 0$ is the catastrophe.