Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2024

4/3/2024: Second Hourly

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

Please write neatly. Use the same page for the answer if possible.

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		100

Problem 1) TF questions (10 points) No justifications are needed.

1) [T] [F] $\int x^7\,dx = 7x^6 + C.$

Solution:
Wrong way. We integrate, not differentiate

2) [T] [F] $\int_{-1}^{1} x^7\,dx = 0.$

Solution:
Cancellation. The function is odd and the left area cancels with the right area.

3) [T] [F] If f is differentiable and $a < b$ are given, then $\int_a^b f'(x)\,dx/(b-a)$ is the average rate of change $(f(b) - f(a))/(b-a)$ of f on $[a, b]$.

Solution:
Yes, by the fundamental theorem and by definition of average rate of change.

4) [T] [F] If $f(x)$ is differentiable, the definite integral $\int_0^1 f(x)\,dx$ can be approximated by Riemann sums.

Solution:
We have defined it as such.

5) [T] [F] If the derivative $g'(x)$ of $g(x) = \int_0^x f(t)\,dt$ is positive for all x, then $f(x)$ is positive for all x.

Solution:
The derivative of the integral is $f(x)$ and because the integral is monotonically increasing, this derivative is positive or zero.

6) [T] [F] If f is differentiable, then $\int_a^b (-f(x))\,dx = -\int_a^b f(x)\,dx$.

Solution:
Yes, we can take the $-$ sign outside the integral.

7) [T] [F] The fundamental theorem of calculus implies $\int_a^b f(x)\,dx = f'(b) - f'(a)$ if f is a differentiable function.

Solution:
Wrong.

8) [T] [F] The family $f_c(x) = c(x-1)^4$ experiences a catastrophe at $c = 0$.

Solution:
The nature of the critical point $x = 1$ changes when deforming c.

9) [T] [F] The function $x\ln(x)$ is an anti-derivative of $\ln(x)$.

Solution:
Differentiate the right hand side to check. It is almost right but not quite.

10) [T] [F] If f is continuous, then $\int_a^b f(-x)\,dx = -\int_a^b f(x)\,dx$.

Solution:
This is not true in general. It would be true for odd functions.

Problem 2) Theorems (10 points) No justifications are needed.

Fill in the empty box to make a true statement:

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a) $\frac{d}{dx}\int_3^x f(t)\ dt =$ [ ] by the fundamental theorem of calculus.

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b) Applying an iteration step $T(x) = x - f(x)/f'(x)$ is called a Newton step. It is used to get closer to a [ ] of f.

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c) A critical point that is a minimum is also called a [ ] equilibrium.

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d) If f is differentiable and $f(-3) = 0$ and $f(3) = 1$, then there is a point x in $(-3, 3)$ for which $f'(x) =$ [ ].

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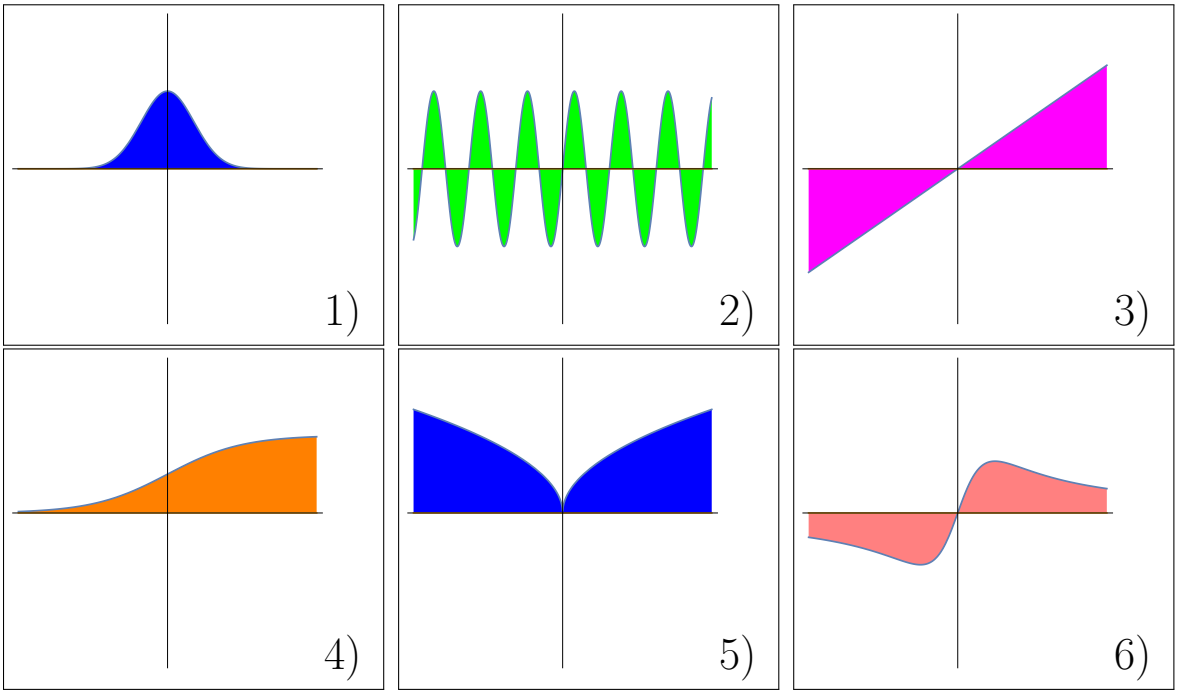
e) Assume $f_c(x)$ is a **family of functions** such that for $c < 10$, there are exactly 7 minima and for $c > 10$ there are exactly 6 minima, then $c = 10$ is called a [ ].

Solution:
a) $f(x)$.
b) Root.
c) stable equilibrium
d) 1/6
e) catastrophe.

Problem 3) Matching (10 points)

Match the following integrals with parts of the regions. No justification is needed.

Integral	Fill in 1-6
$\int_{-4}^{4} \frac{x}{2}\, dx$	
$\int_{-4}^{4} \frac{e^x}{1+e^x}\, dx$	
$\int_{-4}^{4} \sqrt{\lvert x \rvert}\, dx$	
$\int_{-4}^{4} \sin(5x)\, dx$	
$\int_{-4}^{4} e^{-x^2}\, dx$	
$\int_{-4}^{4} \frac{x}{1+x^2}\, dx$	



Solution:
3,4,5,2,1,6.

Problem 4) Chain rule (10 points)

Compute the derivatives $f'(x)$ of the following functions $f(x)$. Each problem is 2 points.

a) $f(x) = \sin(x^9)$

b) $f(x) = e^{\sin(x)}$

c) $f(x) = \cos^5(x)$

d) $f(x) = \tan(\sin(x^2))$

e) $f(x) = \ln(\ln(\ln(x)))$

Solution:

a) $\cos(x^9)9x^8$.

b) $e^{\sin(x)}\cos(x)$.

c) $-5\cos^4(x)\sin(x)$.

d) $\frac{1}{\cos^2(\sin(x^2))}\cos(x^2)2x$.

e) $\frac{1}{\ln(\ln(x))}\frac{1}{\ln(x)}\frac{1}{x}$

Problem 5) Related rates (10 points)

We know that

$$yx^4 + xy^4 + x^2y^2 = 22$$

relates two functions $x = x(t), y = y(t)$ and that $x' = x'(0) = 2$ and $x = x(0) = 2, y = y(0) = 1$.
What is $y' = y'(0)$? [1]

Solution:
$y'x^4 + 4yx^3x' + x'y^4 + 4xy^3y' + 2xy^2x' + 2yx^2y' = 0$. So $16y' + 64 + 2 + 8y' + 8 + 8y' = 0$ This means $32y' + 74 = 0$ and $y' = -74/32 = -37/16$.

[1] As usual, we just always write x, x', y, y' and not $x(0), x'(0), y(0), y'(0)$.

Problem 6) Implicit Differentiation (10 points)

You are given that

$$x^2y^2 - xy + x^8 = 3$$

defines a function $y = y(x)$ near $x = 1, y = 2$. What is $y' = y'(1)$? [2]

Solution:
$2xy^2 + x^2wyy' + 8x^7 - y - xy' = 0$ Plug in $x = 1, y = 2$ and solve for y'. The result is $-14/3$.

[2] As usual, we just always write y, y' and not $y(1), y'(1)$.

Problem 7) Definite integrals (10 points)

Compute the following definite integrals. Each problem is 2 points.

a) $\int_0^1 \frac{1}{x+3}\, dx$

b) $\int_0^1 \frac{2x}{1+x^2}\, dx$

c) $\int_0^1 \frac{2}{1+x^2}\, dx$

d) $\int_0^1 \frac{4x^3}{1+x^4}\, dx$

e) $\int_0^1 7e^{5x+1}\, dx$

Solution:
a) $\ln(4) - \ln(3) = \ln(4/3)$.
b) $\ln(2) - \ln(1) = \ln(2)$
c) $2\arctan(1) - 2\arctan(0) = 2\arctan(1) = \pi/2$
d) $\ln(2) - \ln(1) = \ln(2)$
e) $7(e^6 - e^1)/5$.

Problem 8) Indefinite integrals (10 points)

Solve the indefinite integrals. Each problem is 2 points.

a) $\int 3x^2 + 4x^3 \, dx$

b) $\int 2\cos(x) + 3\sin(x) \, dx$

c) $\int \tan(2x) \, dx$

d) $\int \sin(8x^2)x \, dx$

e) $\int \frac{e^x}{1+e^x} \, dx$

Solution:
a) $x^3 + x^4 + C$
b) $2\sin(x) - 3\cos(x) + C$
c) $-\frac{1}{2}\ln(\cos(2x)) + C$
d) $-\cos(8x^2)/16 + C$
e) $\ln(1 + e^x) + C$.

Problem 9) Newton Step (10 points)

a) (5 points) Do a Newton step

$$T(x) = x - \frac{f(x)}{f'(x)}$$

with the function $f(x) = x^3 - 7 = 0$ starting at $x_0 = 2$.

b) (5 points) Now do a second Newton step.

Solution:
a) $2 - \frac{1}{12} = 23/12$
b) $23/12 - \frac{(23/12)^3 - 7}{3(23/12)^2}$.

Problem 10) Catastrophes (10 points)

Let $f_c(x) = cx^2 - x^2 + x^3$.

a) (3 points) Verify that $x = 0$ is a critical point for all c.

b) (4 points) Determine the stability of this critical point $x = 0$ depending on c.

c) (3 points) For which value of c does a catastrophe appear?

Solution:
a) To find critical points, we differentiate. The derivative is $f_c'(x) = 2cx - 2x + 3x^2$. Independent of c, there is always the critical point $x = 0$ as $f_c'(0) = 0$.
b) The stability is determined by the second derivative $f_c''(x) = 2c - 2 + 6x$ which is $2c - 2$ at $x = 0$. For $c > 1$ this is a minimum, for $c < 1$ this is a maximum.
c) It is at the parameter value $c = 1$, where the number of minima changes. This is called a catastrophe.