

# 5/14/2011: Final exam

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points) No justifications are needed.

- 1)  T  F  $\frac{d}{dx} \log(\cos(x)) = -\tan(x)$ .
- 2)  T  F If  $f'(x) > 0$  for all  $x \geq 0$  and  $f(0) = 1$  then  $f(x) > 0$  for all  $x \geq 0$ .
- 3)  T  F If a function  $f(x)$  has a critical point 1 and  $f''(1) > 0$  then 1 is a local maximum of  $f$ .
- 4)  T  F The anti derivative of  $\log(x)$  is  $1/x$  for all  $x > 0$ .
- 5)  T  F The limit of  $1/(\log|x|)$  for  $x \rightarrow 0$  exists and is 0.
- 6)  T  F Our musical 12 tone scale is based on the exponential function. The frequencies are  $440 \cdot 2^{k/12}$ , where  $k$  is an integer.
- 7)  T  F We have  $\sin(9\pi/4) = -1/\sqrt{2}$ .
- 8)  T  F The function  $t/(e^t - 1)$  has the limit 1 as  $t$  goes to zero.
- 9)  T  F If  $f$  and  $g$  are continuous functions on the real line, then both  $fg$  and  $f/g$  are continuous functions on the real line.
- 10)  T  F The point  $x = 0$  is a critical point of  $f(x) = x^2 - x$ .
- 11)  T  F A function  $f$  for which  $f'(x) < 0$  for all  $x$  is monotonically decreasing.
- 12)  T  F The function  $f(x) = \exp(x)$  has no inflection point.
- 13)  T  F The function  $f(x) = (x^3 - 1)/(x - 1)$  has a limit for  $x \rightarrow 1$ .
- 14)  T  F If the velocity at time  $t$  is  $\sin(t)$  and  $f(0) = 0$ , then the position at time  $t$  is  $f(t) = 1 - \cos(t)$ .
- 15)  T  F If a function  $f$  is differentiable and  $f(x) \rightarrow 0$  for  $x \rightarrow \infty$ , then  $\int_1^\infty f(x) dx$  is bounded.
- 16)  T  F We have the differentiation rule  $(fg)' = (f'g - fg')/g^2$  if  $g$  is not zero.
- 17)  T  F Hopital's rule tells that if  $f(x) \rightarrow 0$  for  $x \rightarrow 0$  and  $g(x) \rightarrow 0$  for  $x \rightarrow 0$ , we have  $\lim_{x \rightarrow 0} f(x)g(x) = \lim_{x \rightarrow 0} f'(x)g'(x)$ .
- 18)  T  F A Newton step for the function  $f$  is  $T(x) = x - \frac{f(x)}{f'(x)}$ .
- 19)  T  F A catastrophe is a critical point of  $f$ .
- 20)  T  F The fundamental theorem of calculus tells that  $\int_0^x f'(t) dt = f(x) - f(0)$  and  $\frac{d}{dx} \int_0^x f(t) dt = f(x)$ .

Problem 2) Matching problem (10 points) No justifications are needed.

a) (5 points) Name dropping: Match results with names

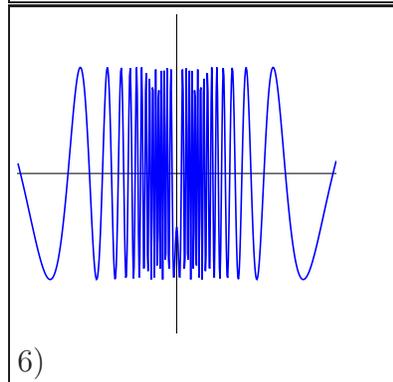
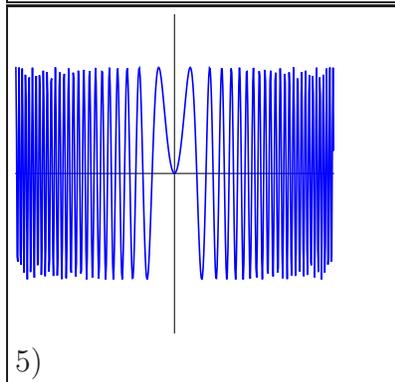
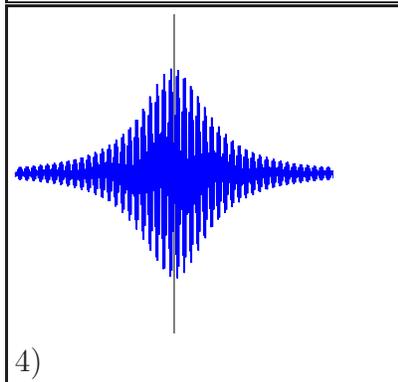
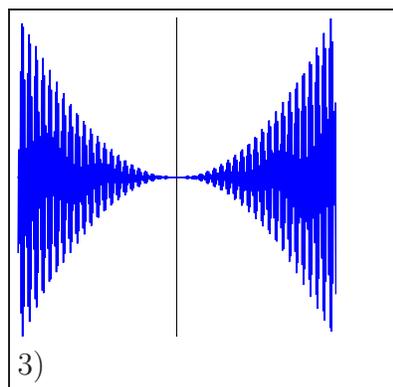
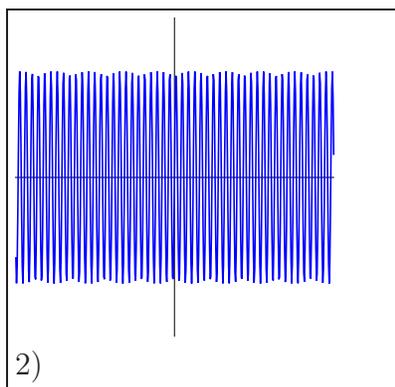
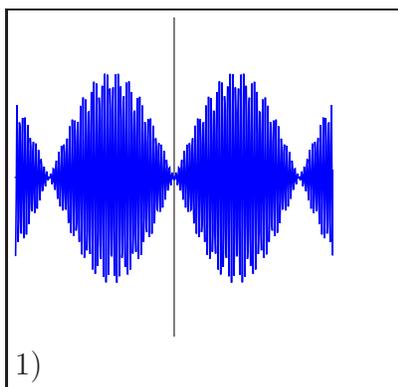
Result	Enter A-G
Fundamental theorem of trigonometry	
Newton steps	
Fundamental theorem of calculus	
Mean value theorem	
Rolle's theorem	
Intermediate value theorem	
Fermat theorem	

A)	$\int_0^1 f'(x) dx = f(1) - f(0)$
B)	$\lim_{x \rightarrow 0} \sin(x)/x = 1$
C)	$f(0) = -1, f(1) = 1$ implies $f(x) = 0$ for some $x \in (0, 1)$ .
D)	$f$ is continuous on $[0, 1]$ then $f$ has a global max and min on $[0, 1]$ .
E)	$T(x) = x - f(x)/f'(x)$ .
F)	If $f(0) = f(1) = 0$ then $f'(x) = 0$ for some $x \in (0, 1)$ .
G)	There exists $x$ in $(0, 1)$ such that $f'(x) = f(1) - f(0)$ .

b) (5 points) Match the functions with their graphs. Mind the ceiling hulls!

Function	Enter 1-6
$\sin(1000x)$	
$\sin(1000x^2)$	
$x^2 \sin(1000x)$	

Function	Enter 1-6
$\sin(1000x/(1+x^2))$	
$\sin(x) \sin(1000x)$	
$\sin(1000x)/(1+x^2)$	



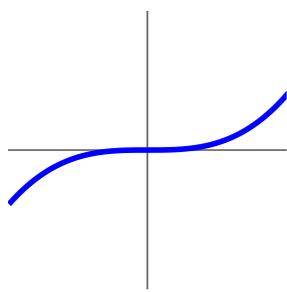
Problem 3) Matching problem (10 points) No justifications are needed.

a) (5 points) Find the relation between the following functions:

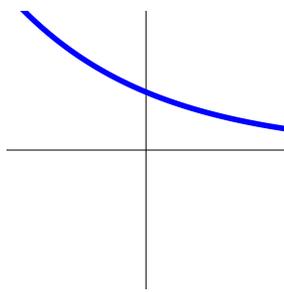
function $f$	function $g$	$f = g'$	$g = f'$	none
$\log(x)$	$1/x$			
$1/x$	$-1/x^2$			
$\tan(x)$	$1/(1+x^2)$			
$\cot(x)$	$-1/\sin^2(x)$			
$\arctan(x)$	$1/\cos^2(x)$			
$\operatorname{arccot}(x)$	$-1/(1+x^2)$			

b) (5 points) Match the following functions (a-d) with their derivatives (1-4) and second derivatives (A-D).

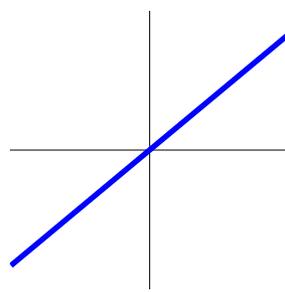
Function a)-d)	Fill in 1)-4)	Fill in A)-D)
graph a)		
graph b)		
graph c)		
graph d)		



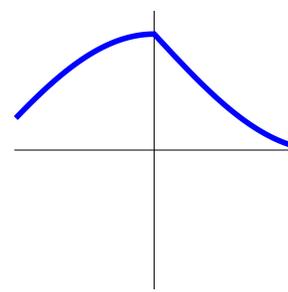
a)



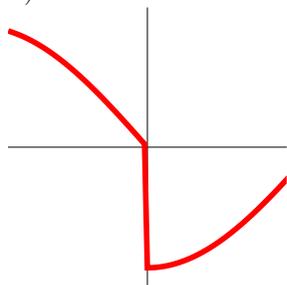
b)



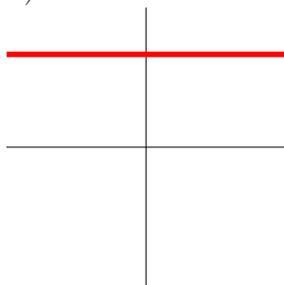
c)



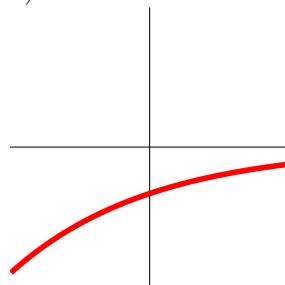
d)



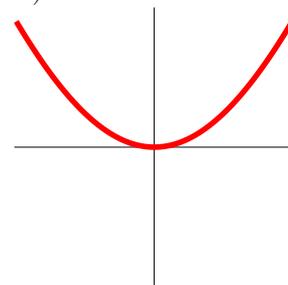
1)



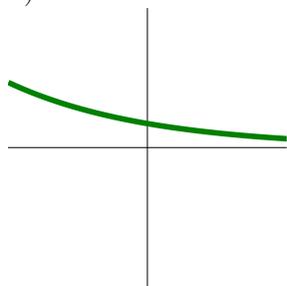
2)



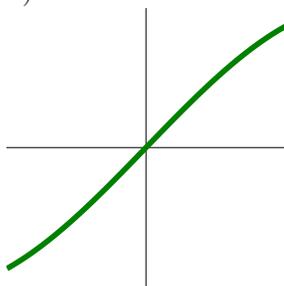
3)



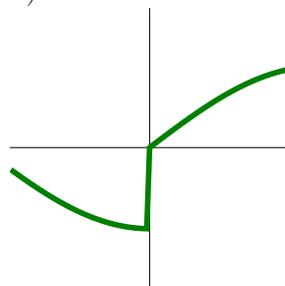
4)



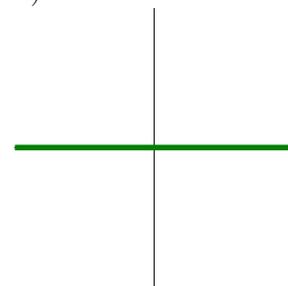
A)



B)

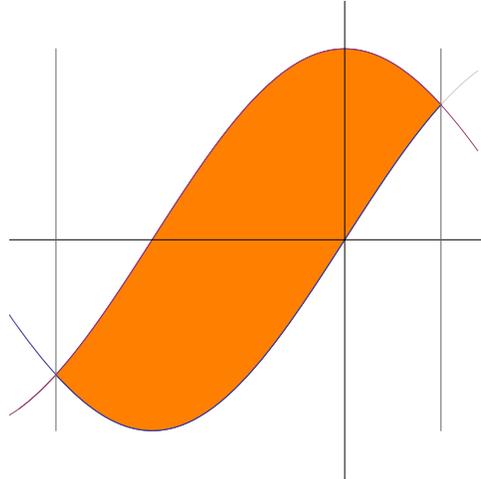


C)



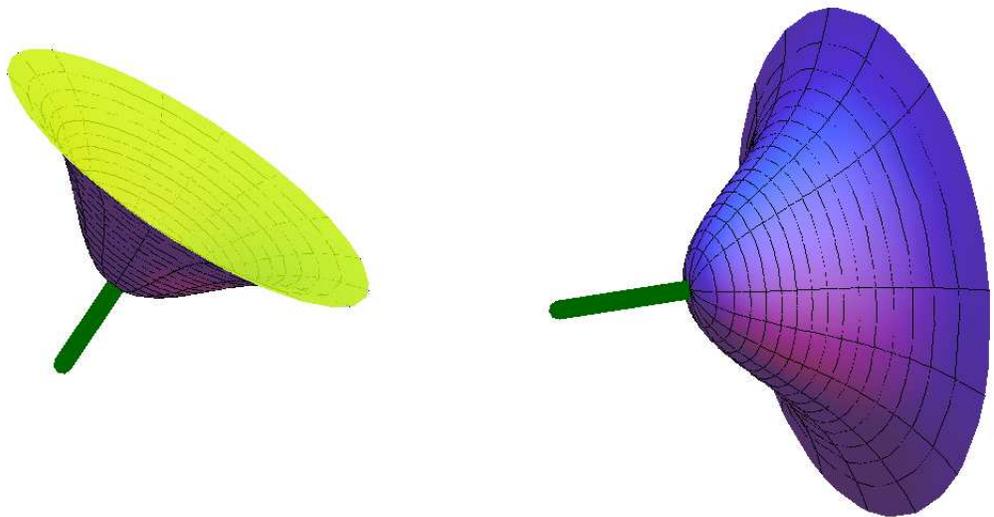
D)

Find the area enclosed by the functions  $f(x) = \sin(x)$  and  $f(x) = \cos(x)$  and between the vertical lines  $x = -3\pi/4$  and  $x = \pi/4$ .



Problem 5) Volume computation (10 points)

If we rotate the graph of the function  $f(x) = \sqrt{\tan(x)}$  from  $x = 0$  to  $x = \pi/4$  we obtain a flower shaped solid. The two pictures below allow to admire it from two sides. Find its volume.



Problem 6) Improper integrals (10 points)

a) (5 points) Find the integral or state that it does not exist

$$\int_1^{\infty} \frac{1}{x^5} dx .$$

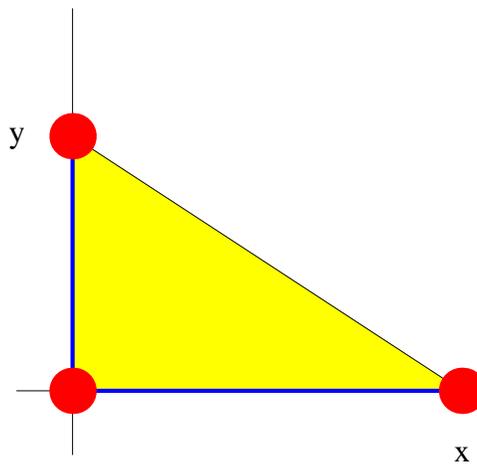
b) (5 points) Find the integral or state that it does not exist

$$\int_0^{\infty} \frac{1}{\cos^2(x)} dx .$$

Problem 7) Extrema (10 points)

a) (5 points) Find the local and global maxima of  $f(x) = 2x^3 - 3x^2$  on the interval  $[-1, 2]$ . Use the second derivative test to check local extrema.

b) (5 points) Which rectangular triangle  $(0, 0), (x, 0), (0, y)$  with  $x + y = 2, x \geq 0, y \geq 0$  has maximal area  $A = xy/2$ ?



Problem 8) Integration by parts (10 points)

Find the antiderivative:

$$\int x^4 \sin(x - 1) dx .$$

Problem 9) Substitution (10 points)

a) (3 points) Solve the integral  $\int \sin^4(x) \cos(x) dx$ .

b) (4 points) Solve the integral  $\int \sqrt{1 + 2x} dx$ .

c) (3 points) Find the integral  $\int \cos(x^2 + 1)x dx$ .

Problem 10) Partial fractions (10 points)

Find

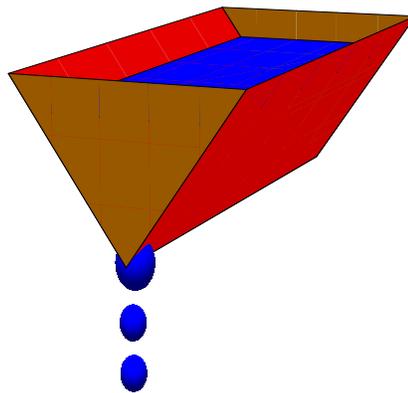
$$\int_1^5 \frac{1}{(x-2)(x-3)} dx .$$

Problem 11) Related rates (10 points)

Water drips from warthog Tuk's bath tub, an uncomfortable container of length 10 for which the width is  $2z$  at height  $z$ . The volume of the water filled up to height  $z$  is

$$V(z) = 10z^2 .$$

If the volume  $V(z(t))$  decreases with constant rate  $V' = -1$ , how fast does the water level sink? Especially, what is  $z'(t)$  at  $t = 1$ ?



Problem 12) Various integration problems (10 points)

Find the anti-derivatives of the following functions:

a) (3 points)  $f(x) = \log(x)/x$ .

b) (4 points)  $f(x) = \frac{1}{x^2-4}$ .

c) (3 points)  $f(x) = \frac{1}{x \log(x)}$ .

Problem 13) Applications (10 points)

If  $F(x) = \log(x)$  is the total cost and  $f(x) = F'(x)$  is the marginal cost and  $g(x) = \log(x)/x$  is the average cost:

a) (4 points) Find the break-even point  $f = g$ .

b) (4 points) Where is the average cost maximal?

c) (2 points) Sweet surprise: why are the results in a) b) the same? The answer to c) is one word.