# Lecture 1: What is Calculus?

Calculus is a powerful tool to describe our world. It formalizes the process of **taking differences** and **taking sums**. Both are natural operations. Differences measure **change**, sums measure how things **accumulate**. We are interested for example in the total amount of precipitation in Boston over a year but we are also interested in how the temperature does change over time. The process of taking differences is in a limit called **derivative**. The process of taking sums is in the limit called **integral**. These two processes are related in an intimate way. In this first lecture, we want to look at these two processes in a discrete setup first, where functions are evaluated only on integers. We will call the process of taking differences a **derivative** and the process of taking sums as **integral**.

Start with the sequence of integers

$$1, 2, 3, 4, \dots .$$

We say $f(1) = 1, f(2) = 2, f(3) = 3$ etc and call $f$ a **function**. It assigns to a number a number. It assigns for example to the number 100 the result $f(100) = 100$. Now we add these numbers up. The sum of the first $n$ numbers is called

$$Sf(n) = f(1) + f(2) + f(3) + \dots + f(n) .$$

In our case we get

$$1, 3, 6, 10, 15, \dots$$

It defines a new function $g$ which satisfies $g(1) = 1, g(2) = 3, g(2) = 6$ etc. The new numbers are known as the **triangular numbers**. From the function $g$ we can get $f$ back by taking difference:

$$Dg(n) = g(n) - g(n - 1) = f(n) .$$

For example $Dg(5) = g(5) - g(4) = 15 - 10 = 5$ and this is indeed $f(5)$.

Finding a formula for the sum $Sf$ is not so easy. The young mathematician **Karl-Friedrich Gauss** realized as a 7 year old kid when giving the task to sum up the first 100 numbers that it is the same as adding up 50 times 101 which is 5050. Gauss found $\boxed{g(n) = n(n + 1)/2}$. He did that by pairing things up. To add up $1 + 2 + 3 + \dots + 10$ for example we can write this as $(1 + 10) + (2 + 9) + (3 + 8) + (4 + 7) + (5 + 6)$ leading to $n/2$ terms of $n + 1$ if $n$ is even. Taking differences again is easier $Dg(n) = (n + 1)n/2 - n(n - 1)/2 = n = f(n)$.

Lets add up the new sequence again and compute $h = Sg$. We get the sequence

$$1, 4, 10, 20, 35, \dots$$

These numbers are called the **tetrahedral numbers** because one use $h(n)$ marbles to build a tetrahedron of side length $n$. For example, we need $h(4) = 20$ golf balls for example to build a tetrahedron of side length 4. The formula which holds for $h$ is $\boxed{h(n) = n(n + 1)(n + 2)/6}$. We see that summing the differences gives the function in the same way as differencing the sum:

$$\boxed{SDf(n) = f(n) - f(0), DSf(n) = f(n)}$$

Don't worry yet, if this is too abstract. We will come back to it again and again. But this is an arithmetic version of the **fundamental theorem of calculus** which we will explore in this course. The process of adding up numbers will lead to the **integral** $\boxed{\int_0^x f(x)\, dx}$. The process of taking differences will lead to the **derivative** $\boxed{\frac{d}{dx} f(x)}$. One of the high lights of this course is to understand the **fundamental theorem of calculus**:

$$\boxed{\int_0^x \tfrac{d}{dt} f(t)\, dt = f(x) - f(0), \quad \tfrac{d}{dx} \int_0^x f(t)\, dt = f(x)}$$

and see why it is such a fantastic result. You see formally that it fits the result for difference and sum. A major goal of this course will be to understand the fundamental theorem result and see its use. But we have packed the essence of the theorem in the above version with $S$ and $D$. It is a version which will lead us.

**1** **Problem:** Given the sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ which satisfies the rule $f(x) = f(x - 1) + f(x - 2)$. It defines a function on the positive integers. For example, $f(6) = 8$. What is the function $g = Df$, if we assume $f(0) = 0$? **Solution:** We take the difference between successive numbers and get the sequence of numbers

$$1, 0, 1, 1, 2, 3, 5, 8, \ldots$$

After 2 entries, the same sequence appears again. We can also deduce directly from the above recursion that $f$ has the property that $\boxed{Df(x) = f(x - 2)}$. It is called the **Fibonnacci sequence**, a sequence of great fame.

**2** **Problem:** Take the same function $f$ given by the sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ but now compute the function $h(n) = Sf(n)$ obtained by summing the first $n$ numbers up. It gives the sequence $1, 2, 4, 7, 12, 20, 33, \ldots$. What sequence is that?
**Solution:** Because $Df(x) = f(x - 2)$ we have $f(x) - f(0) = SDf(x) = Sf(x - 2)$ so that $Sf(x) = f(x + 2) - f(2)$. Summing the Fibonnacci sequence produces the Fibonnacci sequence shifted to the left with $f(2) = 1$ is subtracted. It has been relatively easy to find the sum, because we knew what the difference operation did. This example shows:

$$\boxed{\text{We can study differences to understand sums.}}$$

The next problem illustrates this too:

**3** **Problem:** Find the next term in the sequence
2  6  12  20  30  42  56  72  90  110  132 . **Solution:** Take differences

| 2 | 6 | 12 | 20 | 30 | 42 | 56 | 72 | 90 | 110 | 132 |
|---|---|----|----|----|----|----|----|----|-----|-----|
| 2 | 4 | 6  | 8  | 10 | 12 | 14 | 16 | 18 | 20  | 22  |
| 2 | 2 | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2   | 2   |
| 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0   | 0   |

Now we can add an additional number, starting from the bottom and working us up.

| 2 | 6 | 12 | 20 | 30 | 42 | 56 | 72 | 90 | 110 | 132 | 156 |
|---|---|----|----|----|----|----|----|----|-----|-----|-----|
| 2 | 4 | 6  | 8  | 10 | 12 | 14 | 16 | 18 | 20  | 22  | 24  |
| 2 | 2 | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2   | 2   | 2   |
| 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0   | 0   | 0   |

In the rest of this hour, we talk about some applied and not so applied problems which involve calculus.

# Homework

**1** We have defined $Sf(n) = f(1) + f(2) + ... + f(n)$ and $Df(n) = f(n) - f(n-1)$ and seen

$$f(n) = 1 \qquad \text{we have} \quad g(n) = Sf(n) = n \; .$$
$$f(n) = n \qquad \text{we have} \quad g(n) = Sf(n) = n(n+1)/2.$$
$$f(n) = n(n+1)/2 \quad \text{we have} \quad g(n) = Sf(n) = n(n+1)(n+2)/6.$$

Guess a formula $g(n) = Sf(n)$ for $f(n) = n(n+1)(n+2)/6$ and verify using algebraic manipulation that it satisfies $Dg(n) = f(n)$. Can you see a pattern?

**2** Find the next term in the sequence $3, 12, 33, 72, 135, 228, 357, 528, 747, 1020, 1353....$ To do so, compute successive derivatives $g = Df$ of $f$, then $h = Dg$ until you see a pattern.

**3** The function $f(x) = 2^x$ can first be defined on integers, then on rational numbers like $2^{(3/2)} = \sqrt{2^3}$. We have for example $f(0) = 1, f(1) = 2, f(2) = 4,,...$
a) Verify that $f$ satisfies the equation $Df(x) = f(x-1)$, where $Df(x) = f(x) - f(x-1)$.
b) The function $f(x) = 5^x$ satisfies a similar rule. Which one?

**4** Find $g(n) = Sf(n)$ for the function $f(n) = n^2$. This means we want to find a formula such that $g(1) = 1, g(2) = 5, g(3) = 14$ leading to the sequence of numbers $1, 5, 14, 30, 55, 91, 140, 204, 285, ..$
Note that we have already have computed $Sf$ for $g(n) = n(n+1)/2$ as well as for $h(n) = n$.
Try to write $f$ as a combination of $g$ and $h$ and use the rule $D(f+g) = Df + Dg$.

**5** Find a formula $g(n) = Sf(n)$ for the function $f(n) = 7^n$. First compute the "derivative" $Df$ of $f$ and go from there.

# General remarks about homework

- Make sure to think about the problem yourself first before discussing it with others.

- The time you spend on homework is valuable. Especially the exploration time before you know how to solve it.

- If you do not know how to get started, don't hesitate to ask.