

# Lecture 7: Rate of change

Last week, we defined

$$Df(x) = \frac{f(x+h) - f(x)}{h}$$

It is the **rate of change** of the function with step size  $h$ . When changing  $x$  to  $x+h$  and get a response change  $f(x)$  to  $f(x+h)$ . In this lecture, we take the limit  $h \rightarrow 0$  and derive the important formulas  $\frac{d}{dx} x^n = nx^{n-1}$ ,  $\frac{d}{dx} \exp(x) = \exp(x)$ ,  $\frac{d}{dx} \sin(x) = \cos(x)$ ,  $\frac{d}{dx} \cos(x) = -\sin(x)$  which we have seen already in a discrete setting.

- 1 You walk up a snow hill of height  $f(x) = 30 - x^2$  meters. You walk with a step size of  $h = 0.5$  meters. You are at position  $x = 3$ . How much do you climb or descend when making an other step? We have  $f(3) = 21$  and  $f(3.5) = 17.75$ . We have walked down 3.25 meters. How steep was the snow hill at this point? We have to divide the height difference by the walking distance:  $-3.25/0.5 = -7.5$ .

Today, we take the limit  $h \rightarrow 0$ :

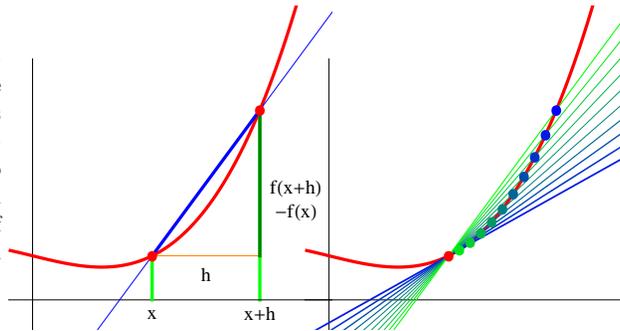
If the limit  $\frac{d}{dx} f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$  exist, we say  $f$  is **differentiable** at the point  $x$ . The value is called the **derivative** or **instantaneous rate of change** of the function  $f$  at  $x$ . We denote the limit also with  $f'(x)$ .

- 2 In the previous problem,  $f(x) = 30 - x^2$  we have

$$f(x+h) - f(x) = [30 - (x+h)^2] - [30 - x^2] = -2xh - h^2$$

Dividing this by  $h$  gives  $-2x - h$ . The limit  $h \rightarrow 0$  gives  $-2x$ . We have just seen that for  $f(x) = x^2$ , we get  $f'(x) = -2x$ . For  $x = 3$ , this is  $-6$ . The actual slope of the snow hill is a bit smaller than the estimate done by walking. The reason is that the hill gets steeper.

The derivative  $f'(x)$  has a geometric meaning. It is the slope of the tangent at  $x$ . This is an important geometric interpretation. It is useful to think about  $x$  as "time" and the derivative as the rate of change of the quantity  $f(x)$  in time.



For  $f(x) = x^n$ , we have  $f'(x) = nx^{n-1}$ .

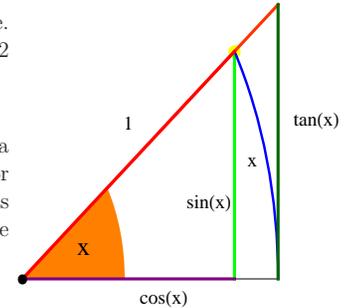
Proof:  $f(x+h) - f(x) = (x+h)^n = (x^n + nx^{n-1}h + a_2h^2 + \dots + h^n) - x^n = nx^{n-1}h + a_2h^2 + \dots + h^n$ . If we divide by  $h$ , we get  $nx^{n-1} + h(a_2 + \dots + h^{n-2})$  for which the limit  $h \rightarrow 0$  exists: it is  $nx^{n-1}$ . This is an important result because most functions can be approximated very well with polynomials.

For  $f(x) = \sin(x)$  we have  $f'(0) = 1$  because the differential quotient is  $[f(0+h) - f(0)]/h = \sin(h)/h = \text{sinc}(h)$ . We have already seen that the limit is 1 before. Lets look at it again geometrically. For all  $0 < x < \pi/2$  we have

$$\sin(x) \leq x \leq \tan(x)$$

- 3 [dividing by 2 squeezes the area of the sector by the area of triangles.] Because  $\tan(x)/\sin(x) = 1/\cos(x) \rightarrow 1$  for  $x \rightarrow 0$ , the value of  $\text{sinc}(x) = \sin(x)/x$  must go to 1 as  $x \rightarrow 0$ . Renaming the variable  $x$  with the variable  $h$ , we see the **fundamental theorem of trigonometry**

$$\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1$$



- 4 For  $f(x) = \cos(x)$  we have  $f'(x) = 0$ . To see this, look at  $f(0+h) - f(0) = \cos(h) - 1$ . Geometrically, we can use Pythagoras  $\sin^2(h) + (1 - \cos(h))^2 \leq h^2$  to see that  $2 - 2\cos(h) \leq h^2$  or  $1 - \cos(h) \leq h^2/2$  so that  $(1 - \cos(h))/h \leq h/2$  and this goes to 0 for  $h \rightarrow 0$ . We have just nailed down an other important identity

$$\lim_{h \rightarrow 0} \frac{1 - \cos(h)}{h} = 0$$

The interpretation is that the tangent is **horizontal** for the  $\cos$  function at  $x = 0$ . We will call this a critical point later on.

- 5 From the previous two examples, we get

$$\cos(x+h) - \cos(x) = \cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x) = \cos(x)(\cos(h) - 1) - \sin(x)\sin(h)$$

because  $(\cos(h) - 1)/h \rightarrow 0$  and  $\sin(h)/h \rightarrow 1$ , we see that  $[\cos(x+h) - \cos(x)]/h \rightarrow -\sin(x)$ .

For  $f(x) = \cos(ax)$  we have  $f'(x) = -a \sin(ax)$ .

- 6 Similarly,

$$\sin(x+h) - \sin(x) = \cos(x)\sin(h) + \sin(x)\cos(h) - \sin(x) = \sin(x)(\cos(h) - 1) + \cos(x)\sin(h)$$

because  $(\cos(h) - 1)/h \rightarrow 0$  and  $\sin(h)/h \rightarrow 1$ , we see that  $[\sin(x+h) - \sin(x)]/h \rightarrow \cos(x)$ .

for  $f(x) = \sin(ax)$ , we have  $f'(x) = a \cos(ax)$ .

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

Like  $\pi$ , the Euler number  $e$  is irrational. Here are the first digits: 2.7182818284590452354. If you want to find an approximation, just pick a large  $n$ , like  $n = 100$  and compute  $(1 + 1/n)^n$ . For  $n = 100$  for example, we see  $101^{100}/100^{100}$ . We only need  $101^{100}$  and then put a comma after the first digit to get an approximation. Interested why the limit exists: verify that the fractions  $A_n = (1 + 1/n)^n$  increase and  $B_n = (1 + 1/n)^{(n+1)}$  decrease. Since  $B_n/A_n = (1 + 1/n)$  which goes to 1 for  $n \rightarrow \infty$ , the limit exists. The same argument shows that  $(1 + 1/n)^{2n} = \exp_{1/n}(x)$  increases and  $\exp_{1/n}(x)(1 + 1/n)$  decreases. The limiting function  $\exp(x) = e^x$  is called the **exponential function**. Remember that if we write  $h = 1/n$ , then  $(1 + 1/n)^{nx} = \exp_h(x)$  considered earlier in the course. We can sandwich the exponential function between  $\exp_h(x)$  and  $(1 + h)\exp_h(x)$ :

$$\exp_h(x) \leq \exp(x) \leq \exp_h(x)(1 + h), \quad x \geq 0.$$

For  $x < 0$ , the inequalities are reversed.

7 Lets compute the derivative of  $f(x) = e^x$  at  $x = 0$ . **Answer.** We have

$$\left(\left(1 + \frac{1}{n}\right)^n - 1\right)n \leq (e^h - 1)/h \leq \left(\left(1 + \frac{1}{n}\right)^{n+1} - 1\right)n$$

Use the binomial formula to see that both the left and right hand side go to 1 if  $n \rightarrow \infty$ . Therefore  $f'(0) = 1$ . The exponential function has a graph which has slope 1 at  $x = 0$ .

8 Now, we can get the general case. It follows from  $e^{x+h} - e^x = e^x(e^h - 1)$  that the derivative of  $\exp(x)$  is  $\exp(x)$ .

$$\text{For } f(x) = \exp(ax), \text{ we have } f'(x) = a \exp(ax).$$

It follows from the properties of taking limits that  $(f(x) + g(x))' = f'(x) + g'(x)$ . We also have  $(af(x))' = af'(x)$ . From this, we can now compute many derivatives

9 Find the slope of the tangent of  $f(x) = \sin(3x) + 5 \cos(10x) + e^{5x}$  at the point  $x = 0$ . **Solution:**  $f'(x) = 3 \cos(3x) - 50 \sin(10x) + 5e^{5x}$ . Now evaluate it at  $x = 0$  which is  $3 + 0 + 5 = 8$ .

Finally, lets mention an example of a function which is not everywhere differentiable.

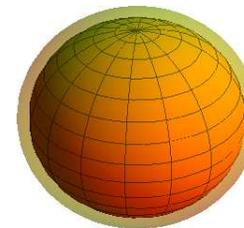
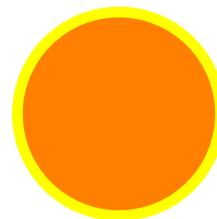
10 The function  $f(x) = |x|$  has the properties that  $f'(x) = 1$  for  $x > 0$  and  $f'(x) = -1$  for  $x < 0$ . The derivative does not exist at  $x = 0$  evenso the function is continuous there. You see that the slope of the graph jumps discontinuously at the point  $x = 0$ .

For a function which is discontinuous at some point, we don't even attempt to differentiate it there. For example, we would not even try to differentiate  $\sin(4/x)$  at  $x = 0$  nor  $f(x) = 1/x^3$  at  $x = 0$  nor  $\sin(x)/|x|$  at  $x = 0$ . Remember these bad guys?

To the end, you might have noticed that in the boxes, more general results have appeared, where  $x$  is replaced by  $ax$ . We will look at this again but in general, the relation  $f'(ax) = af'(ax)$  holds ("if you drive twice as fast, you climb twice as fast").

## Homework

- For which of the following functions does the derivative exist for all points?
  - $|\sin(x)|$
  - $|\exp(x)|$
  - $\exp(x) + \sin(15x)$
  - $|\cos(x)|$
  - $\sin(1/x)$
  - $|\exp(x)| + |1 + \sin(15x)|$
- A circle of radius  $x$  has the area  $f(r) = \pi r^2$ . Find  $\frac{d}{dr}f(r)$ . Can you visualize why this is the same than the circumference of the circle.
  - The sphere of radius  $r$  has the volume  $f(r) = 4\pi r^3/3$ . Find  $\frac{d}{dr}f(r)$  and compare it with the surface area of the sphere.
  - A **hypersphere** of radius  $r$  has the **hyper volume**  $f(r) = \pi^2 r^4/2$ . Find  $\frac{d}{dr}f(r)$ , the volume of the boundary sphere.



- Find the derivatives of the following functions at the point  $x = 2$ .
  - $f(x) = \exp(x) + \sin(x) + x + x^2 + x^3 + x^4 + x^5$ .
  - $f(x) = (x^5 - 1)/(x - 1) + \cos(2x)$ . First heal the function.
  - $f(x) = \frac{1+4x+6x^2+4x^3+x^4}{x^2+2x+1}$ . Dito, first heal!

4 In this problem we compute the derivative of  $\sqrt{x}$  for  $x > 0$ . To do so, we have to find the limit

$$\lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

**Hint:** multiply the top and the bottom with  $(\sqrt{x+h} + \sqrt{x})$  and simplify.

5 A rocket lifts off from Cape Canaveral. The height at time  $t$  is  $h(t) = e^t - 1 + \sqrt{t}$ , at least for the first few seconds. Find the rate of change of the height at time  $t = 1$ . Use the previous problem to get the derivative of  $\sqrt{t}$ .

