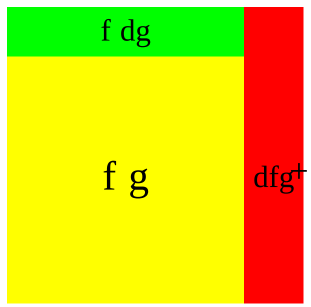
Lecture 9: The product rule

In this lecture, we look at the derivative of a product of functions. The product rule is also called **Leibniz rule** named after Gottfried Leibniz, who found it in 1684. It is a very important rule because it allows us differentiate many more functions. If we wanted to compute the derivative of $f(x) = x\sin(x)$ for example, we would have to get under the hood of the function and compute the limit $\lim(f(x+h) - f(x))/h$. We are too lazy for that. Lets start with the identity

$$f(x+h)g(x+h) - f(x)g(x) = [f(x+h) - f(x)] \cdot g(x+h) + f(x) \cdot [g(x+h) - g(x)]$$



which can be written as $D(fg) = Dfg^+ + fDg$ with $g^+(x) = g(x+h)$. This **quantum Leibniz rule** can also be seen geometrically: the rectangle of area $(f+df)(g+dg)$ is the union of rectangles with area $f \cdot g$, $f \cdot dg$ and $df \cdot g^+$. Divide this relation by h to see

$$\frac{[f(x+h) - f(x)]}{h} \cdot g(x+h) \;\to\; f'(x) \cdot g(x)$$
$$f(x) \cdot \frac{[g(x+h) - g(x)]}{h} \;\to\; f(x) \cdot g'(x) .$$

We get the extraordinarily important **product rule**:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x) .$$

1 Find the derivative function $f'(x)$ for $f(x) = x^3\sin(x)$. **Solution:** We know how to differentiate x^3 and $\sin(x)$ so that $f'(x) = 3x^2\sin(x) + x^3\cos(x)$.

2 While we know

$$\frac{d}{dx}x^5 = 5x^4$$

lets compute this with the Leibniz rule and write $x^5 = x^3 \cdot x^2$. We have

$$\frac{d}{dx}x^3 = 3x^2, \frac{d}{dx}x^2 = 2x .$$

The Leibniz rule gives us $d/dx^5 = 3x^4 + 2x^4 = 5x^4$.

3 Water powered JetLev systems have now gone into production. The water is sucked up from the water surface from a four-inch diameter polyester hose. Consider a system, where the water is carried with you. By Newtons law the force F satisfies $F = p'$, where $p = mv$ is the momentum, the product of your mass and velocity. Written out, this is

$$F(t) = \frac{d}{dt}(m(t)v(t)) .$$

How big is the acceleration v'? The product rule tells us $F = m'v + mv'$ which gives $v' = (F - m'v)/m$. Since we throw out water, $m'(t)$ is negative and $m(t)$ decreases, we accelerate if the force F is kept constant.

The Leibniz rule is also called **product rule**. It suggests a **quotient rule**. One can avoid the quotient rule by writing it as a product $f(x)/g(x) = f(x) \cdot 1/g(x)$ and by using the **reciprocal rule**:

If $g(x) \neq 0$, then
$$\frac{d}{dx}\frac{1}{g(x)} = \frac{-g'(x)}{g(x)^2} .$$

To verify it, stare at the identity

$$\frac{1}{g(x+h)} - \frac{1}{g(x)} = \frac{g(x) - g(x+h)}{g(x)g(x+h)} .$$

Dividing it by h gives $D(1/g(x)) = -Dg(x)/(g(x)g^+(x))$. Taking the limit $h \to 0$ leads to the identity. An other way to derive this is to write $h = 1/g$ and differentiate $1 = gh$ on both sides. The product rule gives $0 = g'h + gh'$ so that $h' = -hg'/g = -g'/g^2$.

4 Find the derivative of $f(x) = 1/x^4$. **Solution**: $f'(x) = -4x^3/x^8 = -4/x^5$. The same computation shows that $\frac{d}{dx}x^n = nx^{n-1}$ holds for all integers n.

The formula $\boxed{\frac{d}{dx}x^n = nx^{n-1}}$ holds for all integers n.

The **quotient rule** is obtained by applying the product rule to $f(x) \cdot (1/g(x))$ and using the reciprocal rule:

If $g(x) \neq 0$, then
$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{[f'(x)g(x) - f(x)g'(x)]}{g^2(x)} .$$

5 Find the derivative of $f(x) = \tan(x)$. **Solution**: because $\tan(x) = \sin(x)/\cos(x)$ we have

$$\tan'(x) = \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} .$$

6 Find the derivative of $f(x) = \frac{2-x}{x^2+x^4+1}$. **Solution**. We apply the quotient rule and get $[(-1)x^2 + x^4 + 1 + (2-x)(2x + 4x^3)]/(x^2 + x^4 + 1)$.

Here are some more problems with solutions:

7 Find the second derivative of $\tan(x)$. **Solution**. We have already computed $\tan'(x) = 1/\cos^2(x)$. Differentiate this again with the quotient rule gives

$$\frac{-\frac{d}{dx}\cos^2(x)}{\cos^4(x)} .$$

We still have to find the derivative of $\cos^2(x)$. The product rule gives $\cos(x)\sin(x) + \sin(x)\cos(x) = 2\cos(x)\sin(x)$. Our final result is

$$2\sin(x)/\cos^3(x) .$$

8 A cylinder has volume $V = \pi r^2 h$, where r is the radius and h is the height. Assume the radius grows like $r(t) = 1+t$ and the height shrinks like $1-\sin(t)$. Does the volume grow or decrease at $t = 0$? **Solution:** The volume $V(t) = \pi(1+t)^2(1-\sin(t))$ is a product of two functions $f(t) = \pi(1+t)^2$ and $g(t) = (1-\sin(t)$. We have $f(0) = 1, g'(0) = 2, f'(0) = 2, g(0) = 1$. The product rule gives gives $V'(0) = \pi 1 \cdot (-1) + \pi 2 \cdot 1 = \pi$. The volume increases in volume at first.

9 On the **Moscow papyrus** dating back to 1850 BC, the general formula $V = h(a^2+ab+b^2)/3$ for a truncated pyramid with base length a, roof length b and height h appeared. Assume $h(t) = 1+\sin(t), a(t) = 1+t, b(t) = 1-2t$. Does the volume of the truncated pyramid grow or decrease at first? **Solution**. We could fill in $a(t), b(t), h(t)$ into the formula for V and compute the derivative using the product rule. A bit faster is to write $f(t) = a^2 + ab + b^2 = (1+t)^2+(1-3t)^2+(1+t)(1-3t)$ and note $f(0) = 3, f'(0) = -6$ then get from $h(t) = (1+\sin(t))$ the data $h(0) = 1, h'(0) = 1$. So that $V'(0) = (h'(0)f(0) - h(0)f'(0))/3 = (1 \cdot 3 - 1(-6))/3 = -1$. The pyramid shrinks in volume at first.

10 We pump up a balloon and let it fly. Assume that the thrust increases like t and the resistance decreases like $1/\sqrt{1-t}$ since the balloon gets smaller. The distance traveled is $f(t) = t/\sqrt{1-t}$. Find the velocity $f'(t)$ at time $t = 0$.

Homework

1 Find the derivatives of the following functions:
a) $f(x) = \sin(3x)\cos(10x)$.
b) $f(x) = \sin^2(x)/x^2$.
c) $f(x) = x^4 \sin(x)\cos(x)$.
d) $f(x) = 1/\sqrt{x}$.
e) $f(x) = \cot(x) + (1+x)/(1+x^2)$.

2 a) Verify that for $f(x) = g(x)h(x)k(x)$ the formula $f' = g'hk + gh'k + ghk'$ holds.
b) Verify the following formula for derivative of $f(x) = g(x)^3$: $f'(x) = 3g^2(x)g'(x)$.

3 a) If $f(x) = \text{sinc}(x) = \sin(x)/x$, find its derivative $g(x) = f'(x)$ and then the derivative of $g(x)$. Then evaluate it at $x = 0$.
b) If you evaluate $g(x)$ at $x = 0$ you obtain $g(0) = f'(0) = 0$. Is the result in a) not a contradiction to the fact that for $g = 0$ the derivative g' is 0?

4 Find the derivative of

$$\frac{\sin(x)}{1 + \cos(x) + \frac{x^4}{1+\cos^2(x)}}$$

at $x = 0$.

5 a) Verify that in general the derivative of $g(x) = f(x)^2$ is $2f(x)f'(x)$.
b) We have already computed the derivative of $f(x) = \sqrt{x}$ in the last homework by directly computing the limit. Lets do it using the product rule. Use part a) of this problem to compute the derivative of

$$g(x) = f(x) \cdot f(x)$$

Use the obtained identity $g'(x) = ...$ to get a formula for $f'(x) = \frac{d}{dx}\sqrt{g(x)}$.
c) Use the same method and the above homework problem 2b) in this homework set to compute the derivative of the cube root function $f(x) = x^{1/3}$.

This last problem 5) is a preparation for the chain rule, we see next Monday. Avoid using the chain rule already here.

Remarks: Like quantum calculus also quantum Leibniz rule is old. . The obove picture explaining the discrete rule (without having to consider any error terms) appears in the article John Dawson, "Wavefronts, BoxDiagrams and the Product Rule: A discovery Approach", 11 Page 102-106, Two Year College Mathematics Journal, 1980.