

## Lecture 10: The chain rule

In this lecture, we look at the derivative of a composition of functions. Also this rule is important. It will allow us to compute derivatives like for  $f(x) = \sin(x^3)$  which is a composition of two functions  $f(x) = x^3$  and  $g(x) = \sin(x)$ . We can in this example not use the product rule since we do not have a product of functions. It is a composition of functions. How do we compute the derivative functions which are "chained" together like this? The answer to this question is given by the **chain rule**:

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

The chain rule follows from the identity

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x) + (g(x+h) - g(x))) - f(g(x))]}{[g(x+h) - g(x)]} \cdot \frac{[g(x+h) - g(x)]}{h}.$$

Write  $H(x) = g(x+h) - g(x)$  in the first part on the right hand side

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x) + H) - f(g(x))]}{H} \cdot \frac{g(x+h) - g(x)}{h}.$$

As  $h \rightarrow 0$ , we also have  $H \rightarrow 0$  and the first part goes to  $f'(g(x))$  and the second factor has  $g'(x)$  as a limit.

1 Find the derivative of  $f(x) = (4x - 1)^{17}$ . **Solution** The inner function is  $4x - 1$  which has the derivative 4. We get therefore  $f'(x) = 17(4x - 1)^6 \cdot 4 = 28(4x - 1)^6$ . Remark. We could have expanded out the power  $(4x - 1)^{17}$  first and avoided the chain rule. Avoiding the **chain rule** is called the **pain rule**.

2 Find the derivative of  $f(x) = \sin(\pi \cos(x))$  at  $x = 0$ . **Solution:** applying the chain rule gives  $\cos(\pi \cos(x)) \cdot (-\pi \sin(x))$ .

3 For linear functions  $f(x) = ax + b, g(x) = cx + d$ , the chain rule can readily be checked. We have  $f(g(x)) = a(cx + d) + b = acx + ad + b$  which has the derivative  $ac$ . Indeed this is the definition of  $f$  times the derivative of  $g$ . You can convince you that the chain rule is true also from this example since if you look closely at a point, then the function is close to linear.

One of the cool applications of the chain rule is that we can compute derivatives of inverse functions:

4 Find the derivative of the natural logarithm function  $\log(x)$ <sup>1</sup>. **Solution** Differentiate the identity  $\exp(\log(x)) = x$ . On the right hand side we have 1. On the left hand side the chain rule gives  $\exp(\log(x)) \log'(x) = x \log'(x) = 1$ . Therefore  $\log'(x) = 1/x$ .

<sup>1</sup>We always write  $\log(x)$  for the natural log. The  $\ln$  notation is old fashioned and only used in obscure places like calculus books and calculators from the last millenium.

$$\frac{d}{dx} \log(x) = 1/x.$$

Denote by  $\arccos(x)$  the inverse of  $\cos(x)$  on  $[0, \pi]$  and with  $\arcsin(x)$  the inverse of  $\sin(x)$  on  $[-\pi/2, \pi/2]$ .

5 Find the derivative of  $\arcsin(x)$ . **Solution.** We write  $x = \sin(\arcsin(x))$  and differentiate.

$$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}.$$

6 Find the derivative of  $\arccos(x)$ . **Solution.** We write  $x = \cos(\arccos(x))$  and differentiate.

$$\frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1-x^2}}.$$

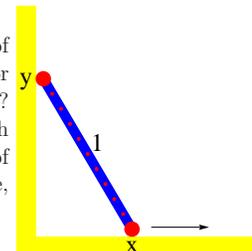
7  $f(x) = \sin(x^2 + 3)$ . Then  $f'(x) = \cos(x^2 + 3)2x$ .

8  $f(x) = \sin(\sin(\sin(x)))$ . Then  $f'(x) = \cos(\sin(\sin(x))) \cos(\sin(x)) \cos(x)$ .

Why is the chain rule called "chain rule"? The reason is that we can chain even more functions together.

9 Lets compute the derivative of  $\sin(\sqrt{x^5 - 1})$  for example. **Solution:** This is a composition of three functions  $f(g(h(x)))$ , where  $h(x) = x^5 - 1$ ,  $g(x) = \sqrt{x}$  and  $f(x) = \sin(x)$ . The chain rule applied to the function  $\sin(x)$  and  $\sqrt{x^5 - 1}$  gives  $\cos(\sqrt{x^5 - 1}) \frac{d}{dx} \sqrt{x^5 - 1}$ . Apply now the chain rule again for the derivative on the right hand side.

Here is the famous **falling ladder problem**. A stick of length 1 slides down a wall. How fast does it hit the floor if it slides horizontally on the floor with constant speed? The ladder connects the point  $(0, y)$  on the wall with  $(x, 0)$  on the floor. We want to express  $y$  as a function of  $x$ . We have  $y = f(x) = \sqrt{1 - x^2}$ . Taking the derivative, assuming  $x' = 1$  gives  $f'(x) = -2x/\sqrt{1 - x^2}$ .



In reality, the ladder breaks away from the wall. One can calculate the force of the ladder to the wall. The force becomes zero at the **break-away angle**  $\theta = \arcsin((2v^2/(3g))^{2/3})$ , where  $g$  is the gravitational acceleration and  $v = x'$  is the velocity.

11 For the brave: find the derivative of  $f(x) = \cos(\cos(\cos(\cos(\cos(\cos(\cos(x)))))$ )).

# Homework

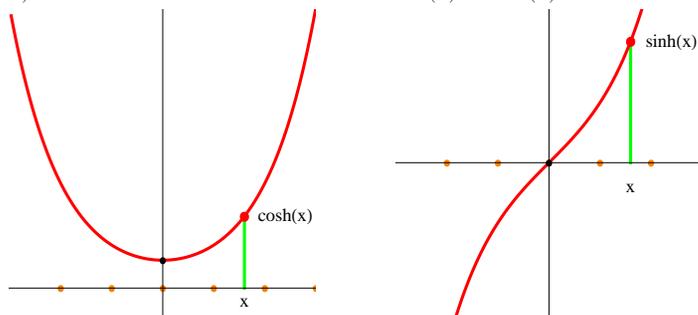
- 1 Find the derivatives of the following functions
 

a) $f(x) = \cos(\sqrt{x})$	c) $f(x) = \exp(1/(1+x))$
b) $f(x) = \tan(1/x^5)$	d) $(2 + \sin(x))^{-5}$
  
- 2 Find the derivatives of the following functions at  $x = 1$ . (Problems c,d) were cut off on the originally distributed pset and are not required. Do them nevertheless to have more practice).
 

a) $f(x) = x^4 \log(x)$ .	c) $(1 + x^2 + x^4)^{100}$
b) $\sqrt{x^5 + 1}$	d) $\frac{5x^4}{2\sqrt{x^5+1}}$
  
- 3 a) Find the derivative of  $f(x) = 1/x$  by differentiating the identity  $xf(x) = 1$ .  
 b) Find the derivative of  $f(x) = \operatorname{arccot}(x)$  by differentiating  $\cot(\operatorname{arccot}(x)) = x$ .
  
- 4 a) Find the derivative of  $f(x) = \sqrt{x}$  by differentiating the identity  $f(x)^2 = x$ .  
 b) Find the derivative of  $f(x) = x^{m/n}$  by differentiating the identity  $f(x)^n = x^m$ .

The function  $f(x) = [\exp(x) + \exp(-x)]/2$  is called  $\cosh(x)$ .  
 The function  $f(x) = [\exp(x) - \exp(-x)]/2$  is called  $\sinh(x)$ .  
 They are called **hyperbolic cosine** and **hyperbolic sine**. The first is even, the second is odd. You can see directly using  $\exp'(x) = \exp(x)$  and  $\exp'(-x) = -\exp(-x)$  that  $\sinh'(x) = \cosh(x)$  and  $\cosh'(x) = \sinh(x)$ . Furthermore  $\exp = \cosh + \sinh$  writes  $\exp$  as a sum of an even and odd function.

- 5 a) Find the derivative of the inverse  $\operatorname{arccosh}(x)$  of  $\cosh(x)$ .  
 b) Find the derivative of the inverse  $\operatorname{arcsinh}(x)$  of  $\sinh(x)$ .



Apropos chain: if you look at the shape of a chain hanging at two points, then it is in the shape of the hyperbolic cosine.

