Lecture 12: Global extrema

In this lecture we look at super maxima. Local maxima are great, global maxima are the greatest. These extrema can occur at critical points of f or at the boundary of the domain, where f is defined.

A point p is called a **global maximum** of f if $f(p) \geq f(x)$ for all x. A point p is called a **global minimum** of f if $f(p) \leq f(x)$ for all x.

How do we find global maxima? We just make a list of all local extrema and boundary points, then pick the largest. Global extrema do not need to exist on the real line. The function $f(x) = x^2$ has a global minimum at $x = 0$ but no global maximum. We can however look at global maxima on finite intervals.

1 Find the global maximum of $f(x) = x^2$ on the interval $[-1, 4]$. **Solution**. We look for local extrema at critical points and at the boundary. Then we compare all these extrema to find the maximum or minimum. The critical points are $x = 0$. The boundary points are $-1, 4$. Comparing the values $f(-1) = 1, f(0) = 0$ and $f(4) = 16$ shows that f has a global maximum at 4 and a global minimum at 0.

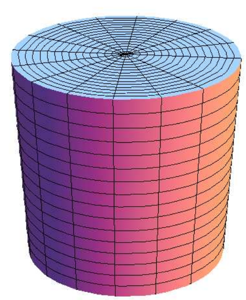
Extreme value theorem A continuous function f on a finite interval $[a, b]$ attains a global maximum and a global minimum.

Here is the argument: Because the function is continuous, the image $f([a, b])$ is a closed interval $[c, d]$. [1] There is a point such that $f(x) = c$, which is a global minimum and a point where $f(x) = d$ which is a global maximum.

Note that the global maximum or minimum can also also on the boundary or points where the derivative dos not exist.

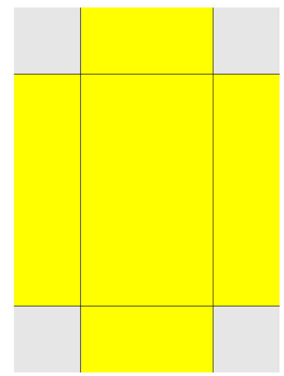
2 Find the global maximum and minimum of the function $f(x) = |x|$. The function has no absolute maximum as it goes to infinity for $x \to \infty$. The function has no critical point on the domain of definition $R \setminus \{0\}$ of the function f'. To see the minimum, we have also to look at the point $x = 0$.

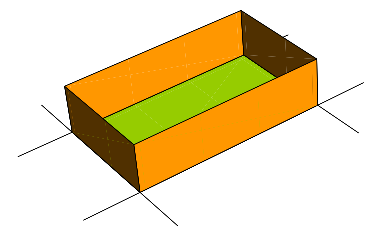
3 A **soda can** is a cylinder of volume $\pi r^2 h$. The surface area $2\pi r h + 2\pi r^2$ measures the amount of material used to manufacture the can. Assume the surface area is 2π, we can solve the equation for $h = (1 - r^2)/r = 1/r - r$ **Solution:** The volume is $f(r) = \pi(r - r^3)$. Find the can with maximal volume: $f'(r) = \pi - 3r^2\pi = 0$ showing $r = 1/\sqrt{3}$. This leads to $h = 2/\sqrt{3}$.



[1]This statement needs more justification but is intuitive enough that we can accept it.

4 Take a **US Letter size** paper of 8×11 inches. [2] If we cut out 4 squares of equal size at the corners, we can fold up the paper to a tray with width $(8 - 2x)$ length $(11 - 2x)$ and height x. Find the $x \in [0, 4]$ for which the volume $f(x) = (8 - 2x)(11 - 2x)x = 4x^3 - 38x^2 + 88x$ is maximal. The solutions to $f'(x) = 12x^2 - 76x + 88 = 0$ are $x = i(19 \pm \sqrt{97})/2$ which is about 1.5 or 5. The second one is larger than 4. We see that



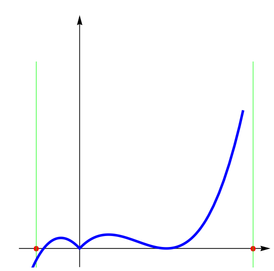


What is the minimal volume? This example illustrates that we might have to look at the boundary of the interval for extrema.

Assume we have a function f which is differentiable except at some points $a_1, \dots, a_n$. We include the end points of the domain of definition in this list. The task is to find the global maximum. How do we proceed?

- 1. Evaluate the function at the point $a_1, .., a_n$.
- 2. Find the local maxima by looking at critical points $b_1, ., ..b_n$.
- 3. Find the maximum of $f(a_1), f(a_2), ..., f(a_n), f(b_1), ...f(b_n)$.

5 Find the global maxima and minima of the function $f(x) = |x| - 2x^2 + x^3$ on the interval $[-1, 2]$.
1) The function is differentiable except at $x = 0$. On $x > 0$ the function is $f(x) = x - 2x^2 + x^3$.
It has derivative $1 - 4x + 3x^2$ which has the root $1/3, 1$.
On $x < 0$ the function is $f(x) = -x - 2x^2 + x^3$, which has a critical point at $(2 - \sqrt{7})/3 = -0.215..$ There is an other critical point but that one is above $x < 0$. So we have the three critical points $1/3, 1, (2 - \sqrt{7})/3$.
2) The function is not differentiable at $x = 0$ and has the boundary points $1, 2$.
3) If we evaluate f at the critical points, we get the values $(0, 0.148, 0, 0.1125, -2, 2)$. The global maximum is at $x = 2$.



[2]The correct size is $17/2 \times 11$ inches we avoid fractions.

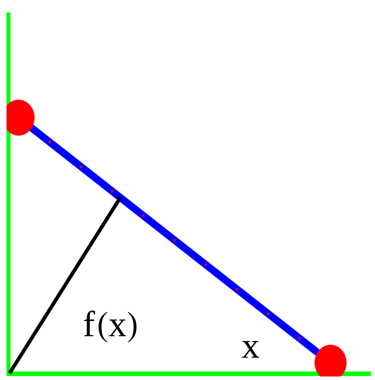
Homework

1 Find the global maxima and minima of the function $f(x) = (x-2)^2$ on the interval $[0,3]$.

2 Find the global maximum and minimum of the function $f(x) = 2x^3 - 3x^2 - 36x$ on the interval $[-5,5]$

3 A candy manufacturer builds spherical candies. Its effectivness is $A(r) - V(r)$, where $A(r)$ is the surface area and $V(r)$ the volume of a candy of radius r. Find the radius, where $f(r) = A(r) - V(r)$ has a global maximum for $r \geq 0$.

4 Lets look at the falling ladder again. But now x denotes the angle, the ladder makes with the floor. Find the angle, where the distance $f(x)$ of the ladder to the wall-floor corner is maximal.
P.S. You can assume the ladder has length 1 but it will not matter how long the ladder is.



5 a) The function $S(p) = -p\log(p)$ is called the **entropy function**. [3] Find the probability $0 < p \leq 1$ which maximizes entropy. One of the most important principle in all science is that nature tries to maximize entropy. In some sense we compute here the number of maximal entropy.
b) We can write $1/x^x = e^{-x\log(x)}$. Find the value x, where x^{-x} has a local maximum which is the point where x^x has a local minimum. . [4]

[3]If $W = 1/p$ is the "Wahrscheinlichkeit", the number of microstates, then $S(p) = p\log(W)$ is the expectation of W, also written just as $\log(W)$. This relation between probability and entropy is inscribed on Bolzmann's thombstone $S = k\log(W)$, where k is an additional constant which depends on units. The Bolzman entropy formula has far reaching consequences to very concrete problems in chemistry.

[4]The identity $a^b = e^{b\log(a)}$ one of the three properties to remember for exponentials. The other two are $a^b a^c = a^{b+c}$ and $(a^b)^c = a^{bc}$.