

Lecture 13: Hopitals rule

The rule

In this lecture, we look at a powerful rule to compute limits. This Hopital's rule works miracles and solves all our remaining worries about limits:

Hopital's rule. If f, g are differentiable and $f(p) = g(p) = 0$ and $g'(p) \neq 0$, then

$$\lim_{x \rightarrow p} \frac{f(x)}{g(x)} = \frac{f'(p)}{g'(p)}.$$

Lets see how it works:

- 1 Lets prove the fundamental theorem of trigonometry again:

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = \lim_{x \rightarrow 0} \frac{\cos(x)}{1} = 1.$$

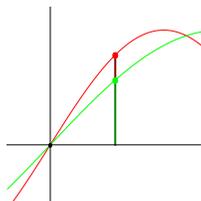
Why did we work so hard for this? Note that we used the fundamental theorem to derive the derivatives for cos and sin at all points. In order to apply l'Hopital, we had to know the derivative. Our work to establish the limit was not in vain.

The proof of the rule is almost comic in its simplicity if we compare it with how fantastically useful it is:

Since $f(p) = g(p) = 0$ we have $Df(p) = f(p+h)/h$ and $Dg(p) = g(p+h)/h$ so that for every $h > 0$ with $g(p+h) \neq 0$ the **quantum l'Hopital rule** holds:

$$\frac{f(p+h)}{g(p+h)} = \frac{Df(p)}{Dg(p)}.$$

Now take the limit $h \rightarrow 0$. The left side is what we want to know, the right side is a quotient of two limits which exist since $g'(p) \neq 0$.¹



Sometimes, we have to administer a medicine twice. To use this, l'Hopital can be improved in that the condition $g'(0) = 0$ can be replaced by the requirement that the limit $\lim_{x \rightarrow p} f'(x)/g'(x)$ exists. Instead of having a rule which replaces a limit with an other limit (we cure a disease with a new one!) we formulate it in the way how it is actually used. The second derivative case could easily be generalized for higher derivatives. There is no need to memorize this. Just remember that you can check in several times to a hospital.

If $f(p) = g(p) = f'(p) = g'(p) = 0$ then $\lim_{x \rightarrow p} \frac{f(x)}{g(x)} = \frac{f''(p)}{g''(p)}$ if $g''(p) \neq 0$.

- 2 Find the limit $\lim_{x \rightarrow 0} (1 - \cos(x))/x^2$. Remember that this limit had also been pivotal to compute the derivatives of trigonometric functions. **Solution:** differentiation gives

$$\lim_{x \rightarrow 0} \frac{-\sin(x)}{2x}.$$

This limit can be obtained with l'Hopital again.

$$\lim_{x \rightarrow 0} \frac{-\sin(x)}{2x} = \lim_{x \rightarrow 0} \frac{-\cos(x)}{2} = -1/2.$$

- 3 Find the limit $f(x) = (\exp(x^2) - 1)/\sin(x^2)$ for $x \rightarrow 0$.
- 4 What do you get if you apply l'Hopital to the limit $[f(x+h) - f(x)]/h$ as $h \rightarrow 0$?
- 5 Find $\lim_{x \rightarrow \infty} x \sin(1/x)$. **Solution.** Write $y = 1/x$ then $\sin(y)/y$. Now we have a limit, where the denominator and nominator both go to zero.

The case when both sides converge to infinity can be reduced the other case by looking at $A = f/g = (1/g(x))/(1/f(x))$ which has the limit $g'(x)/g^2(x)/f'(x)/f^2(x) = g'(x)/f'(x)((1/g)/(1/f))^2 = g'/f'(f^2/g^2) = (g'/f')A^2$, so that $A = f'(p)/g'(p)$. We see:

If $\lim_{x \rightarrow p} f(x) = \lim_{x \rightarrow p} g(x) = \infty$ for $x \rightarrow p$ and $g'(p) \neq 0$, then

$$\lim_{x \rightarrow p} \frac{f(x)}{g(x)} = \frac{f'(p)}{g'(p)}.$$

- 2 What is the limit $\lim_{x \rightarrow 0} x^x$? This answers the intriguing question: what is **What is 0⁰?**

Solution: Because $x^x = e^{x \log(x)}$, it is enough to understand the limit $x \log(x)$ for $x \rightarrow 0$.

$$\lim_{x \rightarrow 0} \frac{\log(x)}{1/x}.$$

Now the limit can be seen as the limit $(1/x)/(-1/x^2) = -x$ which goes to 0. Therefore $\lim_{x \rightarrow 0} x^x = 1$. (We assume $x > 0$ to have real values x^x)

- 3 Find the limit $\lim_{x \rightarrow 2} \frac{x^2 - 4x + 4}{\sin^2(x-2)}$. **Solution:** this is a case where $f(2) = f'(2) = g(2) = g'(2) = 0$ but $g''(0) = 2$. The limit is $f''(2)/g''(2) = 2/2 = 1$.

Hopital's rule always works in calculus situations, where functions are differentiable. The rule can fail if differentiability of f or g fails. Here is an other "rare" example:

- 4 **Deja Vue:** Find $\frac{\sqrt{x^2+1}}{x}$ for $x \rightarrow \infty$. L'Hopital gives $x/\sqrt{x^2+1}$ which in terms gives again $\frac{\sqrt{x^2+1}}{x}$. Apply l'Hopital again to get the original function. We got an infinite loop. If the limit is A , then the procedure tells that it is equal to $1/A$. The limit must therefore be 1. This case can be covered easily without l'Hopital: divide both sides by x to get $\sqrt{1+1/x^2}$. Now, we can see the limit 1.

Scarecrow: given $f(x) = x \sin(1/x^4)e^{-1/x^2}$ and $g(x) = e^{-1/x^2}$. What is the limit $f(x)/g(x)$ as $x \rightarrow 0$. **Solution.** Since the functions f and g are not differentiable at $x = 0$ l'Hopital is not appropriate. The example appears in textbooks because the limit still exists. Look at $f/g = x \sin(1/x^4)$ which satisfies $|f(x)/g(x)| \leq |x|$ and converges to 0 for $x \rightarrow 0$.



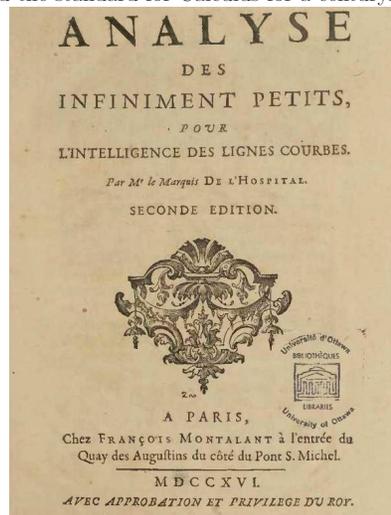
¹Some books refer to the intermediate value theorem here. This is not necessary.

²It appears in <http://mathworld.wolfram.com/LHopitalsRule.html>

- 6 Given a differentiable function satisfying $g(0) = 0$. Verify that the limit $\lim_{x \rightarrow 0} f(g(x))/g(x)$ is $f'(0)$. **Solution:** You check in the homework that the result is $f'(g(0))$.

History

The rule appeared in the "first calculus book" the world has known. The book with name "Analyse des Infiniment Petits pour l'intelligence des Lignes Courbes" appeared in 1696 and was written by Guillaume de l'Hopital, a text if typeset in a modern font would probably fit onto 50-100 pages.³ It is now clear that the mathematical content of Hopital's book is mostly due to **Johannes Bernoulli** who became a mathematical "mercenary" for l'Hopital: *Clifford Truesdell write in his article "The New Bernoulli Edition",⁴ about this "most extraordinary agreement in the history of science": l'Hopital wrote: "I will be happy to give you a retainer of 300 pounds, beginning with the first of January of this year ... I promise shortly to increase this retainer, which I know is very modest, as soon as my affairs are somewhat straightened out ... I am not so unreasonable as to demand in return all of your time, but I will ask you to give me at intervals some hours of your time to work on what I request and also to communicate to me your discoveries, at the same time asking you not to disclose any of them to others. I ask you even not to send here to Mr. Varignon or to others any copies of the writings you have left with me; if they are published, I will not be at all pleased. Answer me regarding all this ..."* Bernoulli's response is lost, but a letter from l'Hopital indicates that it was quickly accepted. From this point on, Bernoulli was a "giant enchained" (Truesdell). Clifford Truesdell also mentions that the book of l'Hopital has remained the standard for Calculus for a century.



Homework

- 1 For the following functions, find the limits as $x \rightarrow 0$:
- $(x^2 - x)/\sin(x)$
 - $(\exp(x) - 1)/(\exp(3x) - 1)$
 - $\sin^3(3x)/\sin^3(5x)$
 - $x + \log(x)x + \frac{\sin(x^2)}{\sin^2(x)}$
 - $\sin(\sin(\sin(\exp(\sin(x)))))/\sin(\sin(\exp(\sin(x))))$.
- 2 For the following functions, find the limits as $x \rightarrow 1$:
- $(x^2 - x - 1)/(\cos(x - 1) - 1)$
 - $(\exp(x) - e)/(\exp(3x) - e^3)$
- For the following functions, find the limits as $x \rightarrow \infty$:
- $(x^2 - x - 1)/\sqrt{x^4 + 1}$
 - $(x - 4)/(4x + \sin(x) + 8)$
- 3 Here is an FUD attempt on l'Hopital's rule: Define $f(x) = x + \cos(x)\sin(x)$ and $g(x) = \exp(\sin(x))(x + \cos(x)\sin(x))$.
- Show that $f'(x)/g'(x)$ converges to zero as $x \rightarrow \infty$.
 - Verify that $f(x)/g(x)$ remains in the interval $[1/e, e]$ but **does not converge**. The function is not differentiable at ∞ . There is no problem with l'Hopital.
- 4 Take the same functions from the previous example and look at the limit $f(x)/g(x)$ for $x \rightarrow 0$. Now things are nice and dandy because the functions are differentiable at 0.
- 5 a) Assume a function $f(x)$ satisfies $f(0) = 0$ and $f'(0) \neq 0$. Verify the following formula

$$\lim_{x \rightarrow 0} f(ax)/f(bx) = a/b.$$

- b) Given a differentiable function g satisfying $g(0) = 0$ and a differentiable function f . Verify that

$$\lim_{x \rightarrow 0} \frac{f(g(x))}{g(x)} = f'(0).$$

³Stewart's book with 1200 pages probably contains about 4 million characters, about 12 times more than l'Hopital's book. It also contains more material of course. The OCRed text of l'Hopital's book of 200 pages has 300'000 characters.

⁴Isis, Vol. 49, No. 1, 1958, pages 54-62