Lecture 14: Newton's method

In the intermediate value theorem lecture, we have seen a simple method to find a root of a function: start with an interval $[a,b]$ such that $f(a) < 0$ and $f(b) > 0$, then successively half the interval always choosing the side on which the function takes different signs at the boundary. We are then $(b-a)/2^n$ close to the root in n steps. If the function is differentiable we can do much better and use the value of the derivative at a boundary point to get closer. If we draw a tangent at $(x, f(x))$, then

$$f'(x) = \frac{f(x)}{x - T(x)} .$$

because $f'(x)$ is the slope of the tangent and the right hand side is "rise" over "run". If we solve for $T(x)$ we get

The **Newton map** is defined as

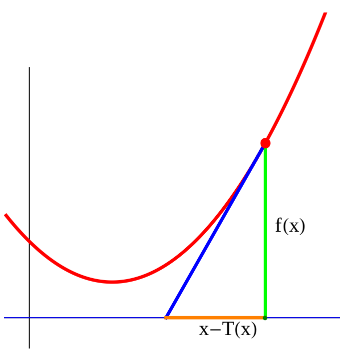
$$T(x) = x - \frac{f(x)}{f'(x)} .$$

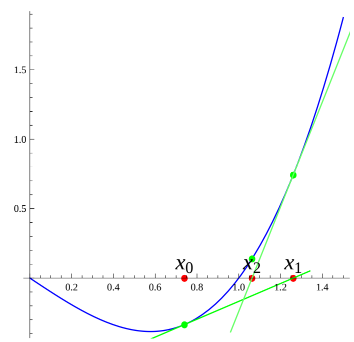
Newton's method is the process to apply this map again and again until we are sufficiently close to the root. It is an extremely fast method to find the root of a function. Start with a point x, then compute a new point $x_1 = T(x)$, where

$$T(x) = x - f(x)/f'(x) .$$

Now iterate this again and again.

If p is a root such that $f'(p) \neq 0$, and x_0 is close to p, then $x_1 = T(x), x_2 = T^2(x)$ converges to the root p.





1 If $f(x) = ax + b$, we reach the root in one step.

2 If $f(x) = x^2$ then $T(x) = x - x^2/(2x) = x/2$. We get exponentially fast to the root 0 but not as fast as the method promises. Indeed, the root is also a critical point which slows us down.

3 The Newton map brings us to infinity if we start at a critical point.

Newton used the method to find the roots of polynomials. The method is so fast that it amazes: Starting 0.1 close to the point, we have after one step 0.01 after 2 steps 0.0001 after 3 steps 0.00000001 and after 4 steps 0.0000000000000001.

The Newton method converges extremely fast to a root $f(p) = 0$ if $f'(p) \neq 0$ if we start sufficiently close to the root.

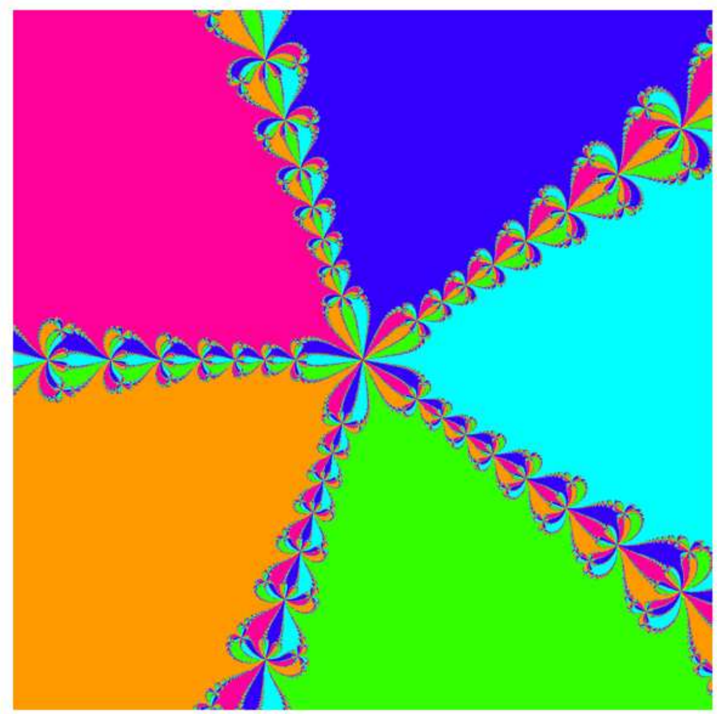
In 10 steps we can get a $2^{10} = 1024$ digits accuracy. Having a fast method to compute roots is useful. For example in computer graphics, where things can not be fast enough. Also in number theory, when working with integers having thousands of digits the Newton method can help. Besides that, there is theoretical use which can explain for example the stability of planetary motion.

4 Verify that the Newton map $T(x)$ in the case $f(x) = (x-1)^3$ has the property that we approach the root $x = 1$. **Solution**. You see that the approach is not that fast: we get $T(x) = x + (1-x)/3 = (1+2x)/3$. It converges exponentially fast, but not superexponential. T The reason is that the derivative at $x - 1$ is not zero. That slows us down.

If we have several roots, and we start at some point, to which root will the Newton method converge? Doe it at all converge? This is an interesting question. It is also historically intriguing because it is one of the first cases, where "chaos" can be observed at the end of the 19'th century.

5 Find the Newton map in the case $f(x) = x^5 - 1$. **Solution** $T(x) = x - (x^5 - 1)/(5x^4)$.

If we look for roots in the complex like for $f(x) = x^5 - 1$ which has 5 roots in the complex plane, the basin of attraction of each of the points is a complicated set, a so called **Newton fractal**. Here is the picture:



6 Lets compute $\sqrt{2}$ to 12 digits accuracy - by hand! We want to find a root $f(x) = x^2 - 2$. The Newton map is $T(x) = x - (x^2 - 2)/(2x)$. Lets start with $x = 1$.

$$\begin{aligned} T(1) &= 1 - (1-2)/2 = 3/2 \\ T(3/2) &= 3/2 - ((3/2)^2 - 2)/3 = 17/12 \\ T(17/12) &= 577/408 \\ T(577/408) &= 665857/470832 \end{aligned}$$

This is already $1.6 \cdot 10^{-12}$ close to the real root!

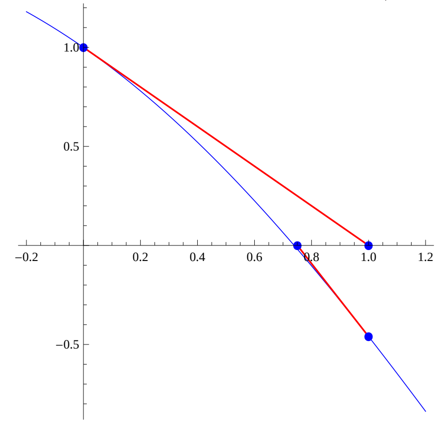
7 To find the cube root of 10 we have to find a root of $f(x) = x^3 - 10$. The Newton map is $T(x) = x - (x^3 - 10)/(3x^2)$. If we start with $x = 2$, we get the following steps $2, 13/6, 3277/1521, 105569067476/49000820427$. After three steps we have a result which is already $2.2 \cdot 10^{-9}$ close to the root.

The Newton method is an incredibly fast algorithm to get roots x_0 of equations. Simply scrumtrulescent.

Homework

1 Find the Newton map $T(x) = x - f(x)/f'(x)$ in the following cases
a) $f(x) = x^3$
b) $f(x) = e^x$
c) $f(x) = e^{-x^2}$ d) $f(x) = 2\tan(x)$.

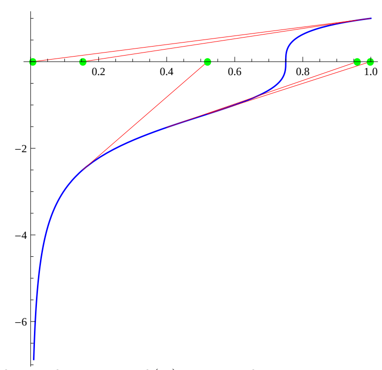
2 a) The sinc function $f(x) = \sin(x)/x$ has a root between 1 and 4. We get closer to the root by doing a Newton step starting with $x = \pi/2$. Do this step



3 The Newton map is handy to compute square roots. Assume we cant to find the square root of 99. We have to solve $\sqrt{99} = x$ or $f(x) = x^2 - 99 = 0$. Perform two Newton steps $T(x) = x - (x^2 - 99)/(2x)$ starting at $x = 10$.

4 a) Find the Newton step $T(x) = x - f(x)/f'(x)$ in the case $f(x) = 1/x$ and $f(x) = x^6$.
b) Find the Newton step $T(x)$ in general if $f(x) = x^\alpha$, where α is a real number.

5 A chaotic Newton map. Verify that the Newton map in the case $f(x) = (4 - 3/x)^{1/3}$ is the quadratic map $T(x) = 4x(1-x)$. We will see a demonstration in class which shows that this map is a true random number generator. The Newton map does not converge.



The graph of the function $f(x)$ and a few Newton steps. The function is continuous on $(0, 1)$. Its derivative too except at $x = 2/3$.